3.2.3 Block Diagram of Differential Equation Models

A *mathematical block diagram* gives a graphically representation of a mathematical model. The block diagram in itself gives good information of the structure of the model, e.g. how subsystems are connected. Furthermore, block diagram models can be simulated directly in simulation tools such as SIMULINK and LabVIEW.

Figure 3.1 shows the most frequently used blocks, which we can call the elementary blocks, that we use in drawing block diagrams. the is described below.

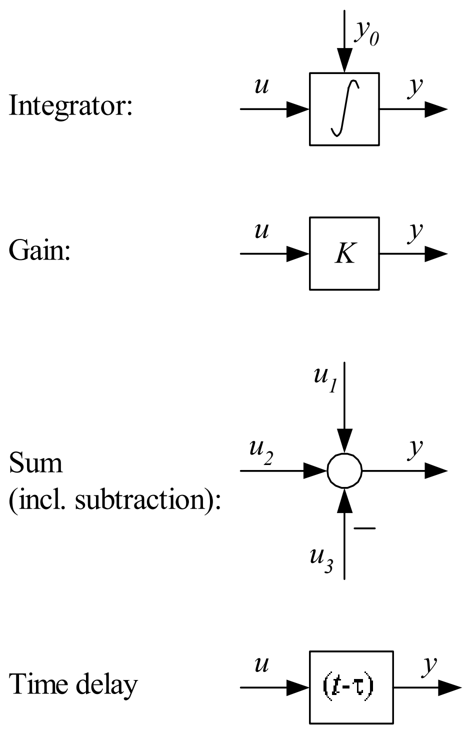

Figure 3.1: Elementary blocks for drawing block diagrams

- **Integrator block**: The output (variable) y of the integrator is equal to the time-integral of the input (variable) u, plus the initial value $y(t=0)$ of the output:

$$y(t) = y(0) + \int_0^t u(\theta)\, d\theta \tag{3.28}$$

- **Gain block**: The relation between the input u and the output y is

$$y(t) = Ku(t) \tag{3.29}$$

where the gain K is any number. The name "gain" is used even if K actually has an absolute value less than 1, that is, even if the gain block actually performs an attenuation.

- **Sum block**: The output y is equal to the sum of the inputs. A negative sign indicates that a subtraction is made:

$$y(t) = u_1(t) + u_2(t) - u_3(t) \tag{3.30}$$

A pluss-sign or no sign indicates that the signal (or variable) enters the block positively. The number of inputs to the sum block is free.

- **Time-delay block**: This block expresses that the output is equal to the input delayed time τ:

$$y(t) = u(t - \tau) \tag{3.31}$$

We will now work through a simple example to be familiar with the procedure of developing block diagrams. We will use all the blocks described above. We will draw a block diagram for the model

$$a_1\dot{x}(t) + a_0x(t) = bu(t - \tau) \tag{3.32}$$

which is a first order linear differential equation for x with a time-delayed input u. The initial state is $x(0)$. We regard x as the output variable. We want the block diagram to show the solution $x(t)$ of the differential equation (3.32). Therefore, before we start drawing, we will express $x(t)$ as the solution to (3.32): From (3.32) we get

$$\dot{x}(t) = \frac{1}{a_1}\left[-a_0x(t) + bu(t - \tau)\right] \tag{3.33}$$

which we integrate (on both sides) from time 0 to t (θ is here used as the integration variable):

$$\int_0^t \{\dot{x}(\theta)\}\, d\theta = x(t) - x(0) = \int_0^t \frac{1}{a_1}\left[-a_0x(\theta) + bu(\theta - \tau)\right]\, d\theta \tag{3.34}$$

which gives

$$x(t) = x(0) + \int_0^t \underbrace{\frac{1}{a_1}\left[-a_0x(\theta) + bu(\theta - \tau)\right]}_{\dot{x}(\theta)} d\theta \tag{3.35}$$

We will use (3.35) as the starting point for drawing the block diagram. For the drawing we need the following blocks: An integrator, three gains blocks (for $a_0 x$, $bu(t-\tau)$ and the multiplication of the parenthesis with the factor $1/a_1$), a time-delay block for the time-delay of u, and a sum block for the additive terms in the integrand. First we draw the integrator, then we draw the rest of the block diagram in accordance with the expression for $x(t)$ as given by (3.35). Figure 3.2 shows the resulting block diagram.

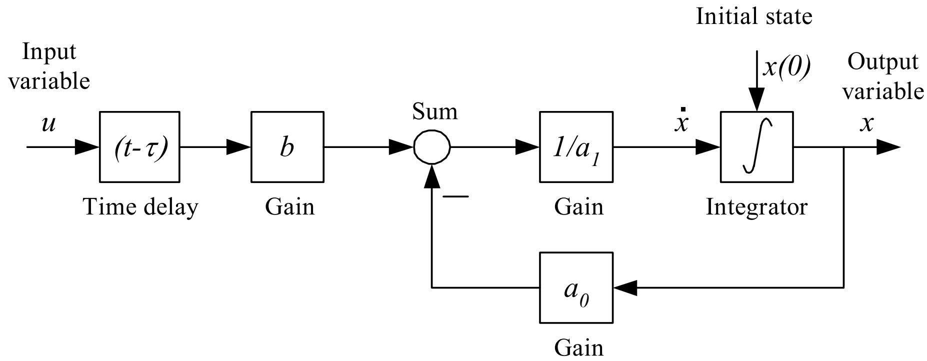

Figure 3.2: The block diagram corresponding to (3.35)

In the example above the differential equation is of first order, so we need only one integrator in the block diagram. The following example shows how we can draw a block diagram for a differential equation of higher order, here: order two. The trick is to find an equivalent state-space model (which consists of a set of first order differential equations), and then draw a block diagram for this state-space model.

Example 14 ***Block diagram for a second order differential equation***

Given the differential equation

$$m\ddot{y} = -D\dot{y} - K_f y + F \tag{3.36}$$

(which is a model of the mass-spring-damper system, cf. Example 4 on page 26). We will draw a block diagram for this differential equation. A systematic procedure is to start writing the differential equation as a state-space model and then draw a block diagram for this state-space model. We found a state-space model on page 53, namely (3.13), (3.14).

The model is repeated here:

$$\dot{x}_1 = x_2 \tag{3.37}$$

$$\dot{x}_2 = \frac{1}{m}\left(-K_f x_1 - D x_2 + u\right) \tag{3.38}$$

By integrating these differential equations we get the following expressions (solutions) for the state-variables $x_1(t)$ and $x_2(t)$:

$$x_1(t) = x_1(0) + \int_0^t [x_2(\theta)]\, d\theta \tag{3.39}$$

$$x_2(t) = x_2(0) + \int_0^t \frac{1}{m}\left[-K_f x_1(\theta) - D x_2(\theta) + u(\theta)\right] d\theta \tag{3.40}$$

from which we draw the block diagram. First we draw an integrator for x_1 and an integrator for x_2, and then we draw the rest of the block diagram according to the model. The resulting block diagram is shown in Figure 3.3.

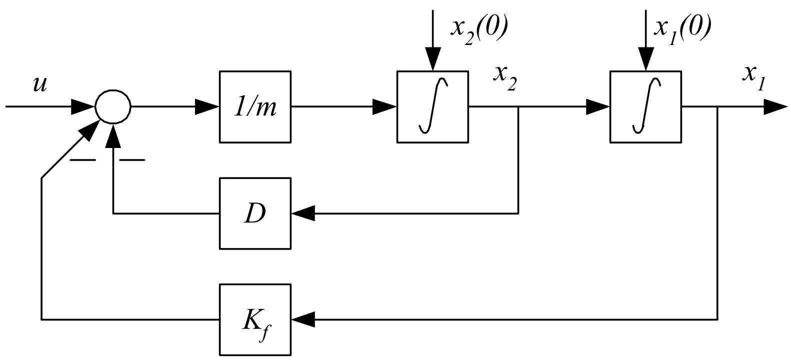

Figure 3.3: Block diagram of (3.39) – (3.40)

[End of Example 14]

Other (non-linear) blocks

When drawing block diagrams, you can use other blocks than the elementary blocks shown in Figure 3.2 to represent for example non-linear functions. Figure 3.4 shows a few such blocks, but you can define the function and the look of a block yourself.

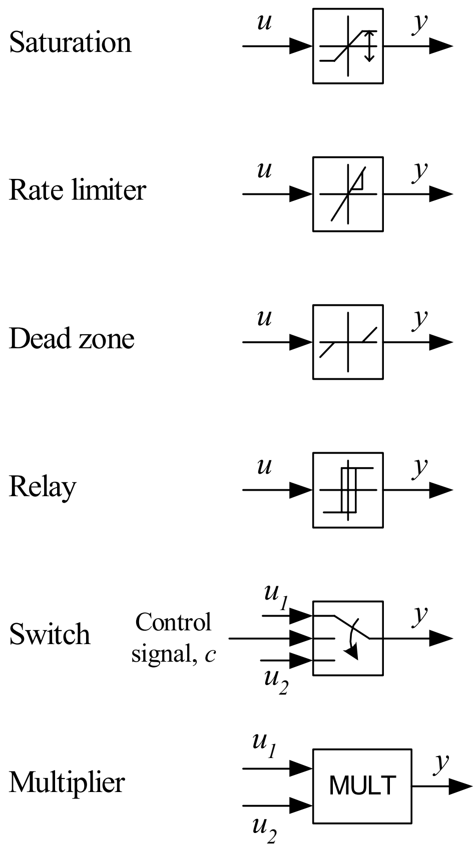

Figure 3.4: Blocks for non-linear functions

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