Article:
Frequency response

Finn Haugen
TechTeach

22. August 2009

Preface

This article gives an introduction to frequency response of continuous-time systems, and it can be downloaded for free from http://techteach.no. It can be used as supplementary material to the book *Basic Dynamics and Control*, which can be purchased from http://techteach.no.

1 Introduction

The *frequency response* of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals – at least periodical signals – which excite systems, as the reference (setpoint) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of *frequency components*. Each frequency component is a sinusoidal signal having a certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system.

The frequency response is an important tool for analysis and design of signal filters (as lowpass filters and highpass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this chapter.

The definition of the frequency response – which will be given in the next section – *applies only to linear models*, but this linear model may very well be the local linear model about some operating point of a non-linear model.

The frequency response can found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.

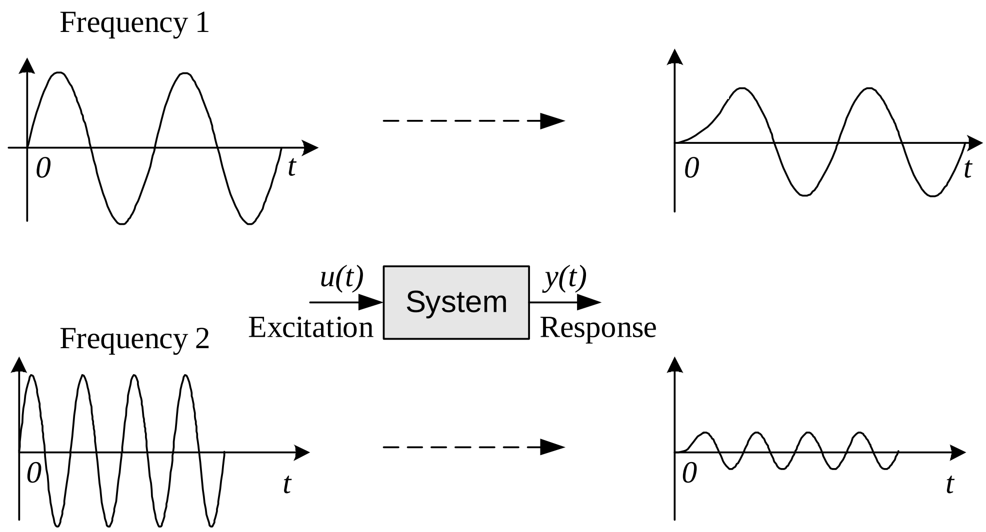

Figure 1: Sinusoidal signals in the input and the resulting responses on the output for two different frequencies

2 How to calculate frequency response from sinusoidal input and output

We can find the frequency response of a system by exciting the system with a sinusoidal signal of amplitude A and frequency ω [rad/s] and observing the response in the output variable of the system. [1] Mathematically, we set the input signal to

$$u(t) = U \sin \omega t \tag{1}$$

See Figure 1. This input signal will give a transient response (which will die, eventually) and a *steady-state* response, $y_s(t)$, in the output variable:

$$\begin{aligned} y_s(t) &= Y \sin(\omega t + \phi) && (2) \\ &= \underbrace{UA}_{Y} \sin(\omega t + \phi) && (3) \end{aligned}$$

[1] The correspondance between a given frequency ω in rad/s and the same same frequency f in Hz is $\omega = 2\pi f$.

Here A is the *(amplitude)gain*, and ϕ (phi) is *the phase lag* in radians. The frequency of $y_s(t)$ will be the same as in $u(t)$. Figure 2 shows in detail $u(t)$ and $y(t)$ for a simulated system. The system which is simulated is

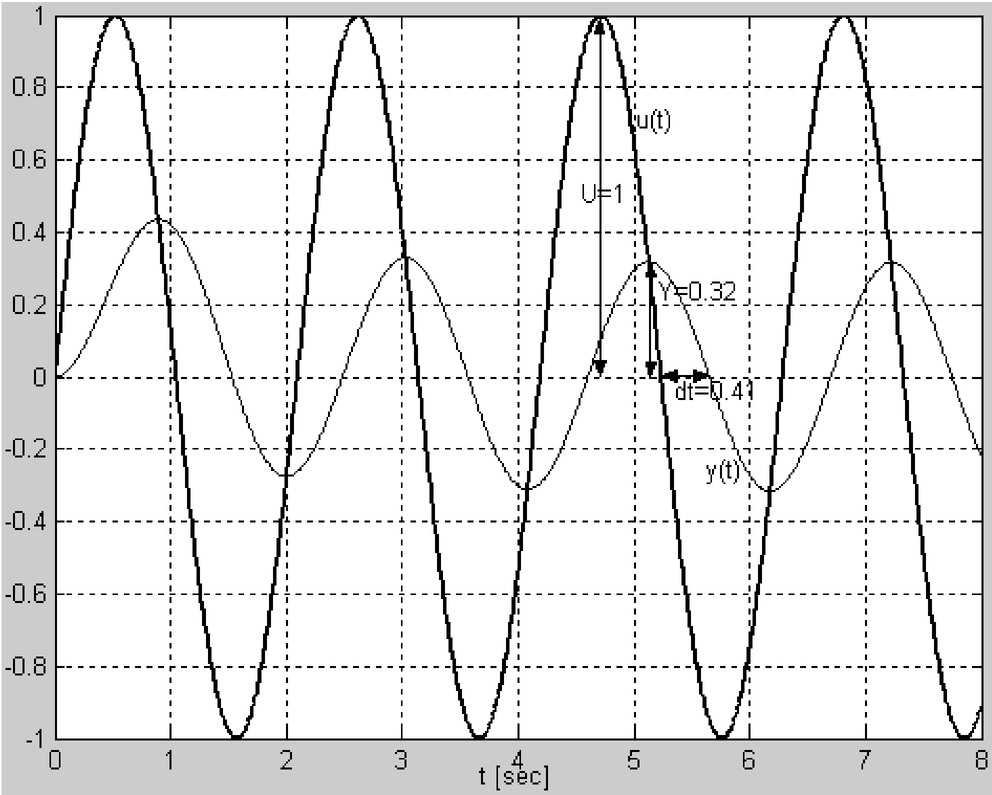

Figure 2: The input signal $u(t)$ and the resulting (sinusoidal) response $y(t)$ for a simulated system. $u(t)$ has frequency $\omega = 3$ rad/s and amplitude $U = 1$. The system is given by (4).

$$y(s) = \frac{1}{s+1} u(s) \tag{4}$$

(a first order system with gain 1 and time-constant 1). The input signal $u(t)$ has frequency $\omega = 3$ rad/s and amplitude $U = 1$.

A is the ratio between the amplitudes of the output signal and the input signal (in steady-state):

$$A = \frac{Y}{U} \tag{5}$$

For the signals shown in Figure 2,

$$A = \frac{Y}{U} = \frac{0.32}{1} = 0.32 \tag{6}$$

ϕ can be calculated by first measuring the time-lag Δt between $u(t)$ and $y_s(t)$ and then calculating ϕ as follows:

$$\phi = -\omega \Delta t \quad \text{[rad]} \tag{7}$$

In Figure 2 we find $\Delta t = 0.41$ sec, which gives

$$\phi = -\omega \Delta t = -3 \cdot 0.41 = -1.23 \text{ rad} \tag{8}$$

The gain A and the phase-lag ϕ are functions of the frequency. We can use the following terminology: $A(\omega)$ is the *gain function*, and $\phi(\omega)$ is the *phase shift function* (or more simply: phase function). We say that $A(\omega)$ and $\phi(\omega)$ expresses the *frequency response* of the system.

3 Bode diagram

It is common to present $A(\omega)$ and $\phi(\omega)$ graphically in a *Bode diagram*, which consists of two subdiagrams, one for $A(\omega)$ and one for $\phi(\omega)$, where the phase values are usually plotted in degrees (not radians). Figure 3 shows a Bode diagram of the frequency response of the system given by (4). The curves may stem from a number of A-values and ϕ-values found in experiments (or simulations) with an sinusoidal input signal of various frequencies. The curves may also stem from the transfer function of the system, as described in Section 4. The frequency axes usually show the 10-logarithm of the frequency in rad/s or in Hz.

Actually, the system (4) is used to generate $u(t)$ and $y(t)$ shown in Figure 2. We have earlier in this chapter calculated $A(3) = 0.32 = -10.2$ dB (the dB-unit is described below) and phase lag $\phi(3) = -1.23$ rad $= -72$ degrees. This gain value and phase lag value are indicated in the Bode diagram in Figure 3.

The $A(\omega)$-axis is usually drawn with decibel (dB) as unit. The decibel value of a number x is calculated as

$$x \text{ [dB]} = 20 \log_{10} x \tag{9}$$

Table 1 shows some examples of dB-values.

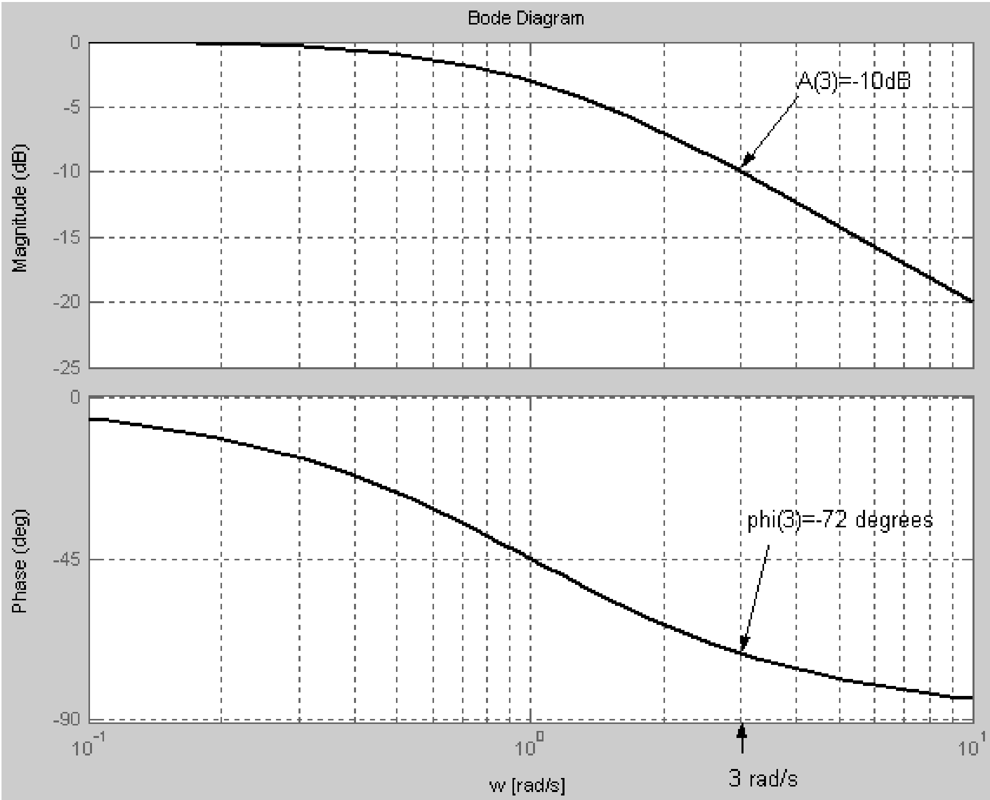

Figure 3: The frequency response of the system given by (4) presented in a Bode diagram

4 How to calculate frequency response from transfer functions

In Section 2 we saw how to find the frequency response from experiments on the system. No model was assumed. However, if we know a transfer function model of the system, we can *calculate the frequency response from the transfer function*, as explained below.

Suppose that system has the transfer function $H(s)$ from input u to output y, that is,

$$y(s) = H(s)u(s) \tag{10}$$

By setting

$$s = j\omega \tag{11}$$

(j is the imaginary unit) into $H(s)$, we get the complex quantity $H(j\omega)$,

0	$=$	$-\infty$dB
0.01	$=$	-40dB
0.1	$=$	-20dB
0.2	$=$	-14dB
0.25	$=$	-12dB
0.5	$=$	-6dB
$\frac{1}{\sqrt{2}}$	$=$	-3dB
1	$=$	0dB
$\sqrt{2}$	$=$	3dB
2	$=$	6dB
$\sqrt{10}$	$=$	10dB
4	$=$	12dB
5	$=$	14dB
10	$=$	20dB
100	$=$	40dB

Table 1: Some dB-values

which is the *frequency response* (function). The *gain function* is

$$A(\omega) = |H(j\omega)| \tag{12}$$

and the *phase shift function* is the angle or argument of $H(j\omega)$:

$$\phi(\omega) = \arg H(j\omega) \tag{13}$$

(The formulas (12) and (13) can be derived using the Laplace transform.)

Eksempel 1 ***Frequency response calculated from a transfer function***

We will find the frequency response for the transfer function

$$H(s) = \frac{K}{Ts + 1} \tag{14}$$

The frequency response becomes

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{K}{Tj\omega + 1} = \frac{K}{\underbrace{1}_{\text{Re}} + j\underbrace{T\omega}_{\text{Im}}} \tag{15}$$

which we write on polar form:

$$\begin{aligned} H(j\omega) &= \frac{K}{\sqrt{1^2 + (T\omega)^2} e^{j \arctan\left(\frac{T\omega}{1}\right)}} && (16) \\ &= \frac{1}{\sqrt{1 + (T\omega)^2}} \, e^{j[-\arctan(T\omega)]} && (17) \\ &= |H(j\omega)| \, e^{j \arg H(j\omega)} && (18) \end{aligned}$$

Thus, the gain function is

$$|H(j\omega)| = \frac{K}{\sqrt{1 + (T\omega)^2}} \tag{19}$$

and the phase function is

$$\arg H(j\omega) = -\arctan(T\omega) \quad \text{[rad]} \tag{20}$$

Figure 4 shows the curves of $|H(j\omega)|$ and $\arg H(j\omega)$ drawn in a Bode diagram. The numerical values along the axes assume $K = 1$ and $T = 1$. (The asymptotes indicated in the figure are not explained in this document.)

To illustrate the use of (19) and (20), let us calculate the gain and phase lag values for the frequency $\omega = 3$ rad/s. We assume that $K = 1$ and $T = 1$. (19) gives

$$|H(j3)| = \frac{1}{\sqrt{1 + 3^2}} = \frac{1}{\sqrt{10}} = 0.316 = -20 \log_{10}\left(\frac{1}{\sqrt{10}}\right) = -10.0 \text{ dB} \tag{21}$$

(20) gives

$$\arg H(j3) = -\arctan(3) = -1.25 \text{ rad} = -71.6 \text{ degrees} \tag{22}$$

[End of Example 1]

The next example shows how the frequency response can be found of a transfer function which consists of several factors in the numerator and/or the denominator.

Eksempel 2 ***Frequency response of a (more complicated) transfer function***

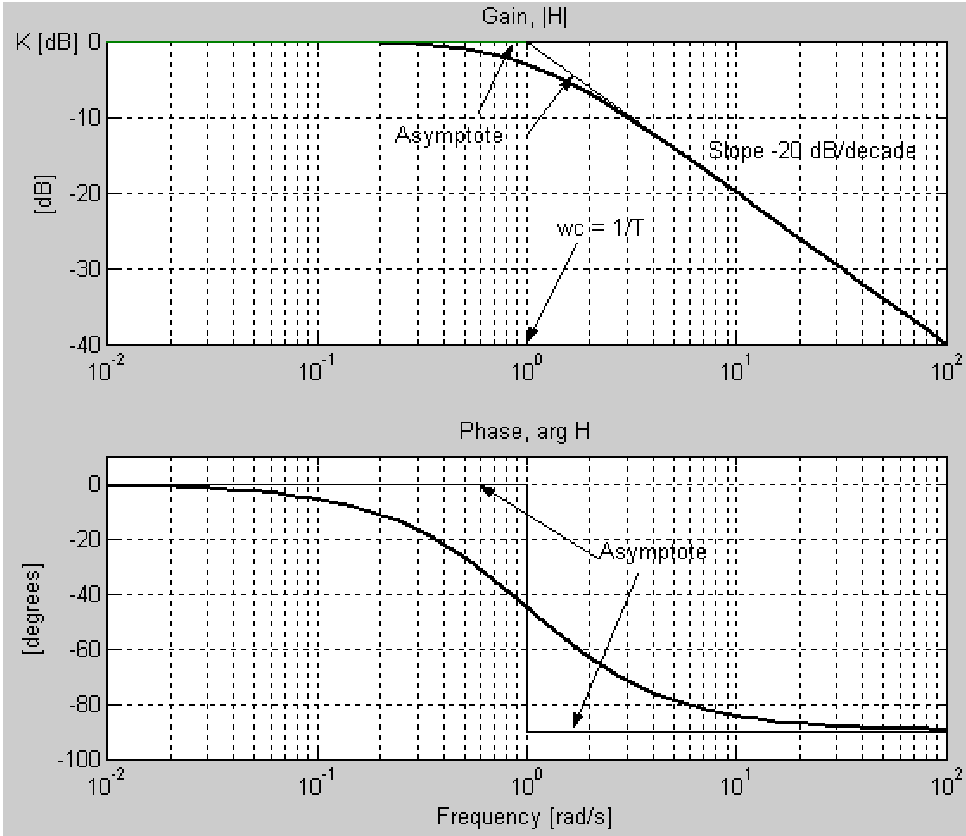

Figure 4: Bode diagram for the frequency response of the first ordens system (14). The asymptotes are not explained in this document.

Given the transfer function

$$H(s) = K\frac{T_1 s + 1}{(T_2 s + 1)\, s} e^{-\tau s} \tag{23}$$

(The term $e^{-\tau s}$ represents a time-delay of τ sec.) We set $s = j\omega$ in $H(s)$ and then sets the individual factors on *polar form*. Finally, we combine these factors so that we end up with a polar form of $H(j\omega)$:

$$H(j\omega) = K\frac{T_1 j\omega + 1}{(T_2 j\omega + 1)\, j\omega}e^{-\tau j\omega} \tag{24}$$

$$= K\frac{\sqrt{1^2 + (T_1\omega)^2}e^{j\arctan\left(\frac{T_1\omega}{1}\right)}}{\left[\sqrt{1^2 + (T_2\omega)^2}e^{j\arctan\left(\frac{T_2\omega}{1}\right)}\right]\left[\sqrt{0^2 + \omega^2}e^{j\frac{\pi}{2}}\right]}e^{-\tau j\omega} \tag{25}$$

$$= \underbrace{\frac{K\sqrt{1 + (T_1\omega)^2}}{\sqrt{1 + (T_2\omega)^2}\omega}}_{|H(j\omega)|} e^{j\underbrace{\left[\arctan(T_1\omega) - \arctan(T_2\omega) - \frac{\pi}{2} - \tau\omega\right]}_{\arg H(j\omega)}} \tag{26}$$

So, the amplitude gain function is

$$A(\omega) = |H(j\omega)| = \frac{K\sqrt{1 + (T_1\omega)^2}}{\sqrt{1 + (T_2\omega)^2}\omega} \tag{27}$$

and the phase shift function is

$$\phi(\omega) = \arg H(j\omega) = \arctan(T_1\omega) - \arctan(T_2\omega) - \frac{\pi}{2} - \tau\omega \tag{28}$$

[End of Example 2]

5 Application of frequency response: Signal filters

5.1 Introduction

A *signal filter* – or just *filter* – is used to attenuate (ideally: remove) a certain frequency interval of frequency components from a signal. These frequency components are typically noise. For example, a lowpass filter is used to attenuate high-frequent components (low-frequent components passes).

Knowledge about filtering functions is crucial in signal processing, but it is useful also in control engineering because control systems can be regarded as filters in the sense that the controlled process variable can follow only a certain range or interval of frequency components in the reference (setpoint) signal, and it will be only a certain frequency range of process disturbances that the control system can compensate for effectively. Furthermore, knowledge about filters can be useful in the analysis and design of physical processes. For example, a stirred tank in a process line can act as a lowpass filter since it attentuates low-frequent components in the inflow to the tank.

In this section we will particularly study *lowpass filters*, which is the most commonly used filtering function, but we will also take a look at *highpass filters*, *bandpass filters* and *bandstop filters*.

Figure 5 shows the gain function for ideal filtering functions and for practical filters (the phase lag functions are not shown). The *passband* is

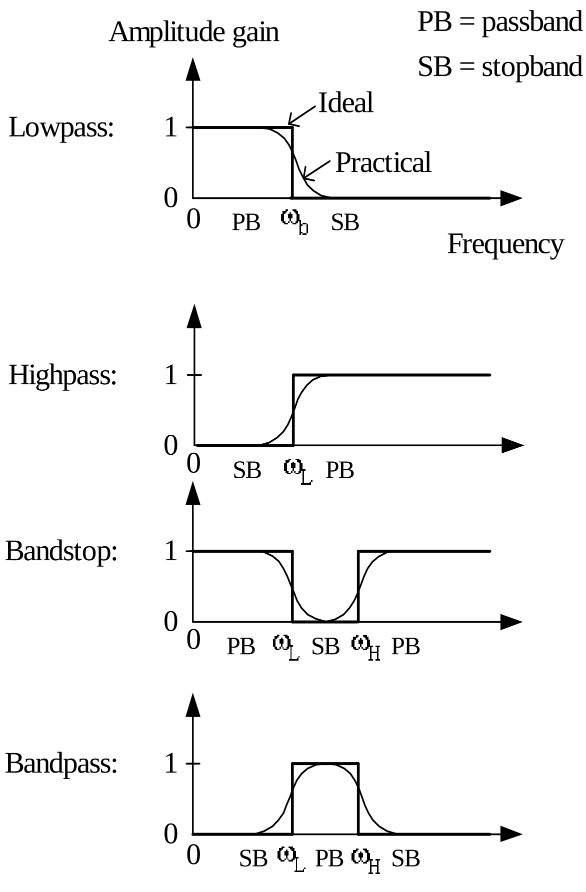

Figure 5: The gain functions for ideal filters and for practical filters of various types.

the frequency interval where the gain function has value 1, ideally (thus, frequency components in this frequency interval passes through the filter, unchanged). The *stopband* is the frequency interval where the gain function has value 0, ideally (thus, frequency components in this frequency interval are stopped through the filter).[2]

It can be shown that transfer functions for ideal filtering functions will have infinitely large order. Therefore, ideal filters can not be realized, neither with analog electronics nor with a filtering algorithm in a computer program.

5.2 First order lowpass filters

The most commonly used signal filter is the first order lowpass filter. As an example, it is the standard measurement filter in a feedback control system.

The transfer function of a first order lowpass filter with input variable u and output variable y is usually written as

$$H(s) = \frac{1}{\frac{s}{\omega_b} + 1} \tag{29}$$

where ω_b [rad/s] is the *bandwidth* of the filter. This is a first order transfer function with gain $K = 1$ and time-constant $T = 1/\omega_b$. The frequency response is

$$\begin{aligned} H(j\omega) &= \frac{1}{\frac{j\omega}{\omega_b} + 1} \qquad (30) \\ &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}\, e^{j \arctan \frac{\omega}{\omega_b}}} \\ &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}}\, e^{j\left(-\arctan \frac{\omega}{\omega_b}\right)} \qquad (31) \end{aligned}$$

The gain function is

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} \tag{32}$$

[2] It is a pity that lowpass filters were not called highstop filters in stead since the main purpose of a lowpass filter is to stop high-frequency components. Similarly, highpass filters should have been called lowstop filters, but it is too late now...

and the phase lag function is

$$\arg H(j\omega) = -\arctan \frac{\omega}{\omega_b} \tag{33}$$

Figure 4 shows exact and asymptotic curves of $|H(j\omega)|$ and $\arg H(j\omega)$ drawn in a Bode diagram. In the figure, $K = 1$ and $\omega_b = \omega_c$.

The bandwidth defines the upper limit of the passband. It is common to say that the bandwidth is the frequency where the filter gain is $1/\sqrt{2} = 0.71 \approx -3$ dB (above the bandwidth the gain is less than $1/\sqrt{2}$). This bandwidth is therefore referred to as the "-3 dB-bandwidth". Now, what is the -3 dB-bandwidth of a first order lowpass filter? It is the ω-solution of the equation

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} = \frac{1}{\sqrt{2}} \tag{34}$$

The solution is $\omega = \omega_b$. Therefore, ω_b [rad/s] given in (29) is the -3 dB-bandwidth in rad/s. In Hertz the bandwidth is

$$f_b = \frac{\omega_b}{2\pi} \tag{35}$$

Figure 6 shows the front panel of a simulator of a first order filter where the input signal consists of a sum of two sinusoids or frequency components of frequency less than and greater than, respectively, the bandwidth. The simulation shows that the low frequent component (0.5 Hz) passes almost unchanged (it is in the passband of the filter), while the high-frequent component (8 Hz) is attenuated (it lies in the stopband).

Eksempel 3 ***The RC-circuit as a lowpass filter***

Figure 7 shows an RC-circuit (the circuit contains the resistor R and the capacitor C). The RC-circuit is frequently used as an analogue lowpass filter: Signals of *low* frequencies *passes* approximately unchanged through the filter, while signals of high frequencies are approximately filtered out (stopped). v_1 is the signal source or input voltage to be filtered, while v_2 is the resulting filtered output voltage.

We will now find a mathematical model relating v_2 to v_1. First we apply the Kirchhoff's voltage law in the circuit which consists the input voltage

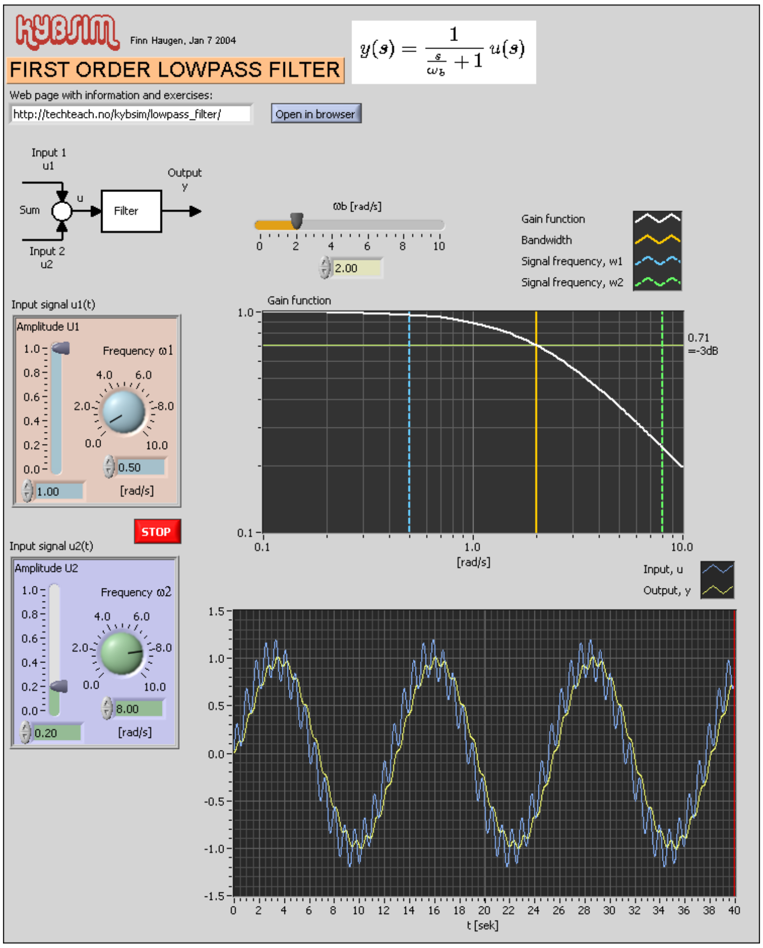

Figure 6: Simulator for a first order lowpass filter where the input signal consists of a sum of two frequency componens

terminals, the resistor, and the capacitor (we consider the voltage drops to be positive clockwise direction):

$$-v_1 + v_R + v_2 = 0 \tag{36}$$

(v_2 equals the voltage drop over the capacitor.) In (36) v_R is given by

$$v_R = Ri \tag{37}$$

We assume that there is no current going through the output terminals. (This is a common assumption, and not unrealistic, since it it typical that the output terminals are connected to a subsequent circuit which has approximately infinite input impedance, causing the current into it to be

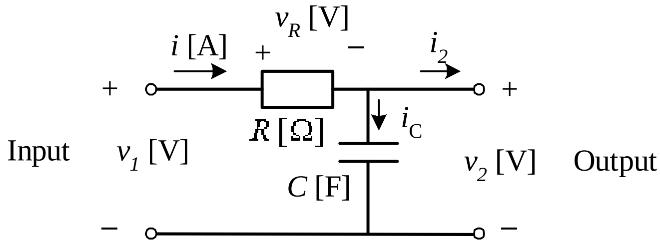

Figure 7: RC-circuit

approximately zero. An operational amplifier is an example of such a load-circuit.) Therefore,

$$i = i_C = C\dot{v}_2 \tag{38}$$

The final model is achieved by using i as given by (38) in (37) and then using v_R as given by (37) for v_R in (36). The model becomes

$$RC\dot{v}_2(t) = v_1(t) - v_2(t) \tag{39}$$

The transfer function from the input voltage v_1 to the output voltage v_2 becomes

$$H_{v_2,v_1}(s) = \frac{1}{RCs + 1} = \frac{1}{\frac{s}{\omega_b} + 1} \tag{40}$$

Thus, the RC-circuit is a first order lowpass filter with bandwidth

$$\omega_b = \frac{1}{RC} \text{ rad/s} \tag{41}$$

If for example $R = 1$ kΩ and $C = 10$ μF, the bandwidth is $\omega_b = 1/RC = 100$ rad/s. (41) can be used to design the RC-circuit (calculate the R- and C-values).

[End of Example 3]