The order should be large enough to include the time-delay. One time step of 0.5 sec corresponds to a time-delay of 0.5 s, and one such time-delay is represented by the factor z^{-1} in the transfer function. The model should therefore include $2/0.5 = 4$ such factors. Hence, the mimimum order of the transfer function is $\underline{\underline{4}}$.

Solution to Exercise 16.7

The model written on the standard LS form:

$$\underline{\underline{\underbrace{\frac{K_T}{R_a}\left[v_a(t_k) - K_e\omega(t_k)\right]}_{y} = \underbrace{\left[\begin{array}{cc} \dot{\omega}(t_k) & 1 \end{array}\right]}_{\varphi} \underbrace{\left[\begin{array}{c} J \\ T_L \end{array}\right]}_{\theta}}} \tag{23.300}$$

$\dot{\omega}(t_k)$ is calculated with the center difference method:

$$\underline{\underline{\dot{\omega}(t_k) \approx \frac{\omega(t_{k+1}) - \omega(t_{k-1})}{2T_s}}} \tag{23.301}$$

where T_s is the time step.

Solution to Exercise 16.8

You can remove the mean values of both the input and the output signals (time series or sequences), and use the devations as new input and output. In more detail: Assume that $\{u(k)\}$ is the original input signal and $\{y(k)\}$ is the output signal, and that m_u and m_y are the respective mean values. The deviation signals are then

$$\{du(k)\} = \{u(k) - m_u\} \tag{23.302}$$

and

$$\{dy(k)\} = \{y(k) - m_y\} \tag{23.303}$$

The signals $\{du(k)\}$ and $\{dy(k)\}$ are used as input and output signals.

Solution to Exercise 17.1

1. L is modeled as

   $$\dot{L}(t) = 0 \tag{23.304}$$

   (17.1) and (23.304) written as a state-space model (the time argument t is omitted for simplicity):

   $$\dot{S} = \overbrace{\frac{1}{T_m}\left[-S + K_m\left(u + L\right)\right]}^{f_1} \tag{23.305}$$

$$\dot{L} = \underbrace{0}_{f_2} \tag{23.306}$$

The general observer formula for continuous-time implementation is

$$\dot{x}_e = f\left(x_e,\, u\right) + Ke \tag{23.307}$$

In detail this becomes

$$\underline{\underline{\dot{S}_e}} = f_1\left(S_e, L_3\right) + K_1 e = \underline{\underline{\frac{1}{T_m}\left[-S_e + K_m\left(u + L_e\right)\right] + K_1 e}} \tag{23.308}$$

$$\underline{\underline{\dot{L}_e}} = f_2\left(S_e, L_3\right) + K_2 e = \underline{\underline{K_2 e}} \tag{23.309}$$

where

$$e = S - S_e = \text{ Speed measurement } - \text{ Speed estimate} \tag{23.310}$$

2. The general observer formula for discrete-time implementation is

$$x_e(t_{k+1}) = x_e(t_k) + T_s\left[f(\cdot, t_k) + Ke(t_k)\right] \tag{23.311}$$

In detail this becomes

$$\underline{\underline{S_e(t_{k+1})}} = S_e\left(t_k\right) + T_s\left[f_1(\cdot, t_k) + K_1 e\left(t_k\right)\right] \tag{23.312}$$

$$= \underline{\underline{S_e\left(t_k\right) + T_s\left(\frac{1}{T_m}\left\{-S_e\left(t_k\right) + K_m\left[u\left(t_k\right) + L_e\left(t_k\right)\right]\right\} + K_1 e\right)}} \tag{23.313}$$

$$\underline{\underline{L_e(t_{k+1})}} = L_e\left(t_k\right) + T_s\left[f_2(\cdot, t_k) + K_2 e\left(t_k\right)\right] \tag{23.314}$$

$$= \underline{\underline{L_e\left(t_k\right) + T_s K_2 e\left(t_k\right)}} \tag{23.315}$$

3. To find the observer gains K_1 and K_2 we need a linearized state-space model on the form

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{23.316}$$

$$\Delta y = C\Delta x + D\Delta u \tag{23.317}$$

Linearization of the state-space model (23.305), (23.306) gives[1]

$$\Delta\dot{S} = \frac{\partial f_1}{\partial S}\Delta S + \frac{\partial f_1}{\partial L}\Delta L + \frac{\partial f_1}{\partial u}\Delta u \tag{23.318}$$

$$= -\frac{1}{T_m}\Delta S + \frac{K_m}{T_m}\Delta L + \frac{K_m}{T_m}\Delta u \tag{23.319}$$

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[1] Since the original model is linear, it is actually not necessary to perform the linearization, but I do it here it anyway to demonstrate the procedure.

$$\begin{aligned}\Delta\dot{L} &= \frac{\partial f_2}{\partial S}\Delta S + \frac{\partial f_2}{\partial L}\Delta L + \frac{\partial f_2}{\partial u}\Delta u && (23.320)\\ &= 0\cdot\Delta S + 0\cdot\Delta L + 0\cdot\Delta u && (23.321)\\ &= 0 && (23.322)\end{aligned}$$

The measurement is S. On matrix-vector form the linearized state-space model is

$$\begin{aligned}\begin{bmatrix}\Delta\dot{S}\\ \Delta\dot{L}\end{bmatrix} &= \begin{bmatrix}\frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial L}\\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial L}\end{bmatrix}\begin{bmatrix}\Delta S\\ \Delta L\end{bmatrix} + \begin{bmatrix}\frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\end{bmatrix}\Delta u && (23.323)\\ &= \underbrace{\begin{bmatrix}-\frac{1}{T_m} & \frac{K_m}{T_m}\\ 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\Delta S\\ \Delta L\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{K_m}{T_m}\\ 0\end{bmatrix}}_{B}\Delta u && (23.324)\end{aligned}$$

$$\Delta S = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\begin{bmatrix}\Delta S\\ \Delta L\end{bmatrix} + \underbrace{\begin{bmatrix}0\end{bmatrix}}_{D}\Delta u \qquad (23.325)$$

The eigenvalues of the observer error dynamics are the roots of the characteristic equation:

$$\begin{aligned}0 &= \det\left[sI - (A - KC)\right] && (23.326)\\ &= \det\left\{\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \left(\begin{bmatrix}-\frac{1}{T_m} & \frac{K_m}{T_m}\\ 0 & 0\end{bmatrix} - \begin{bmatrix}K_1\\ K_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\right\} && (23.327)\\ &= \det\left\{\begin{bmatrix}s + \frac{1}{T_m} + K_1 & -\frac{K_m}{T_m}\\ K_2 & s\end{bmatrix}\right\} && (23.328)\\ &= s^2 + \left(\frac{1}{T_m} + K_1\right)s + \frac{K_2K_m}{T_m} && (23.329)\end{aligned}$$

The second order polynomial (23.329) is to be compared with the second order Butterworth polynomial

$$B_2(s) = (Ts)^2 + 1.4142\,(Ts) + 1 = T^2s^2 + 1.4142Ts + 1 \qquad (23.330)$$

where T is

$$T = \frac{T_r}{n} \qquad (23.331)$$

where T_r is the specified observer response time and $n = 2$ is the order of the observer. Before comparing polynomials we divide (23.330) by T^2 so that it gets the same form as (23.329):

$$B_2^*(s) = s^2 + \frac{1.4142}{T}s + \frac{1}{T^2} \equiv s^2 + \left(\frac{1}{T_m} + K_1\right)s + \frac{K_2K_m}{T_m} \qquad (23.332)$$

Comparing coefficients:

$$\frac{1.4142}{T} = \frac{1}{T_m} + K_1 \tag{23.333}$$

$$\frac{1}{T^2} = \frac{K_2 K_m}{T_m} \tag{23.334}$$

which gives

$$\underline{\underline{K_1 = \frac{1.4142}{T} - \frac{1}{T_m} = \frac{1.4142n}{T_r} - \frac{1}{T_m}}} \tag{23.335}$$

$$\underline{\underline{K_2 = \frac{T_m}{K_m T^2} = \frac{T_m n^2}{K_m T_r^2}}} \tag{23.336}$$

4. The observability test is made with the matrices A and C of the linear state space model. The observability matrix is $(n = 2)$

$$\begin{aligned} M_{\text{obs}} &= \begin{bmatrix} C \\ CA^{2-1} = CA \end{bmatrix} && (23.337) \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \hdashline \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{T_m} & \frac{K_m}{T_m} \\ 0 & 0 \end{bmatrix} \end{bmatrix} && (23.338) \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{T_m} & \frac{K_m}{T_m} \end{bmatrix} && (23.339) \end{aligned}$$

The determinant of M_{obs} is

$$\det\left(M_{\text{obs}}\right) = 1 \cdot \left(\frac{K_m}{T_m}\right) - \left(-\frac{1}{T_m}\right) \cdot 0 = \frac{K_m}{T_m} \tag{23.340}$$

Since $\frac{K_m}{T_m} \neq 0$, the rank of M_{obs} is full (2), and the system is observable.

5. To speed up the response of the estimates, the observer response time T_r can be decreased. This will increase the observer gains, and the estimates become more noisy.

6. To make the estimated become smoother you can let the estimates pass through lowpass filters.

7. The feedforward control function is derived from the motor model, which is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{23.341}$$

In this model, the speed is substituted by its reference or setpoint S_r, the load is substituted by its estimate, and then the equation is solved for the control variable. The result is

$$\underline{\underline{u_f(t) = \frac{1}{K_m} T_m \dot{S}_r(t) + S_r(t) - L_e(t)}} \tag{23.342}$$

In Exercise 18.1 experimental results with feedforward control with this motor are shown. In that exercise a Kalman Filter is used in stead of an observer, but you can expect the same results with an observer.

8. Increased robustness against sensor failure can be obtained as follows: The feedback to the speed controller (e.g. a PI controller) is permanently based on the estimated measurement, S_e, as calculated by an observer. If the sensor is failing (assuming some kind of measurement error detection has been implemented, of course), the estimates are *not updated* by the (erroneous) measurement. This can be implemented by multiplying $K_1 e$ in (23.308) and $K_2 e$ in (23.309) by zero so that the effective continuous-time observer formulas are

$$\dot{S}_e = \frac{1}{T_m} \left[-S_e + K_m \left(u + L_e \right) \right] \tag{23.343}$$

$$\dot{L}_e = 0 \tag{23.344}$$

The discrete-time observer formulas are

$$S_e(t_{k+1}) = S_e\left(t_k\right) + T_s \left(\frac{1}{T_m} \left\{ -S_e\left(t_k\right) + K_m \left[u\left(t_k\right) + L_e\left(t_k\right) \right] \right\} \right) \tag{23.345}$$

$$L_e(t_{k+1}) = L_e\left(t_k\right) \tag{23.346}$$

Hence, the observer is just a *simulator* of the motor. So, during sensor failure, the speed controller acts on basis of the simulated speed, and this is better than acting on basis of an erroneous speed measurement.

In Exercise 18.1 experimental results with sensor failure with this motor are shown. In that exercise a Kalman Filter is used in stead of an observer, but you can expect the same results with an observer.

Solution to Exercise 17.2