20.2 Deriving the control function

The following two sections cover these cases, respectively:

- *All* the state variables are controlled
- *Not all* the state variables are controlled

20.2.1 Case 1: All state variables are controlled

It assumed that the process model is a state space model on the following form:

$$\dot{x} = f(x, v) + B(x, v) \cdot u \tag{20.1}$$

or, simpler,

$$\dot{x} = f + Bu \tag{20.2}$$

x is the state vector, v is the disturbance vector, and u is the control vector. f is a vector of scalar functions, and B is a matrix of scalar functions.

Note that the control vector u is assumed to appear *linearly* in the model.

Assume that the output vector is

$$y = x \tag{20.3}$$

By taking the derivative of (20.3) and using (20.2) we obtain the following differential equation describing the process output vector:

$$\dot{y} = f + Bu \tag{20.4}$$

Assume that r_y is the reference (or setpoint) of y.

With the above assumptions, we derive the control function as follows: We start by defining the *transformed control vector* as

$$z \overset{\text{def}}{=} f + Bu \tag{20.5}$$

Then (20.4) can be written as

$$\dot{y} = z \tag{20.6}$$

which are n decoupled or independent *integrators* (n is the number of state variables), because $y(t) = \int_0^t z \, d\tau$. The transfer function from z to y is

$$\frac{y(s)}{z(s)} = \frac{1}{s} \tag{20.7}$$

We can denote (20.6) as the *transformed process.*

We will now derive the control function for this integrator process, and thereafter derive the final control function. How can you control an integrator? With feedback and feedforward! A good choice for the *feedback controller* is a PI controller (proportional plus integral) because the controller should contain integral action to ensure zero steady-state control error in the presence of unmodelled disturbances (and there are such in a real system). The proportional action is necessary to get a stable control system (if a pure integral controller acts on an integration process the closed loop system becomes marginally stable, i.e. it is pure oscillatory). The multiloop feedback PI controller is

$$z_{\text{fb}} = K_p e + K_i \int_0^t e d\tau \tag{20.8}$$

where e is the control error:

$$e \stackrel{\text{def}}{=} r_y - y \tag{20.9}$$

In (20.8) K_p and K_i are diagonal matrices:

$$K_p = \begin{bmatrix} K_{p_1} & 0 & \cdots & 0 \\ 0 & K_{p_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{p_n} \end{bmatrix} = \text{diag}(K_{p_1}, K_{p_2}, \cdots, K_{p_n}) \tag{20.10}$$

$$K_i = \begin{bmatrix} K_{i_1} & 0 & \cdots & 0 \\ 0 & K_{i_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{i_n} \end{bmatrix} = \text{diag}(K_{i_1}, K_{i_2}, \cdots, K_{i_n}) \tag{20.11}$$

where the scalar values are

$$K_{i_j} = \frac{K_{p_j}}{T_{i_j}} \tag{20.12}$$

where K_{p_j} is the proportional gain and T_{i_j} is the integral time of control loop no. j. K_{p_j} and T_{i_j} can be calculated in several ways. *Skogestad's method* is one option. Skogestad's method is reviewed in Appendix A. From Table A.1 (the second row) we get, since $\tau = 0$ and $K = 1$,

$$K_{p_j} = \frac{1}{T_{C_j}} \tag{20.13}$$

and

$$T_{i_j} = cT_{C_j} \tag{20.14}$$

where T_{C_j} is the specified time constant of feedback loop no. j, and c is a coefficient that can be set to e.g. 1.5, cf. Appendix A.

In addition to the PI feedback action the controller should contain *feedforward* from the reference r_y to get fast reference tracking when needed (assuming the reference is varying). The feedforward control function can be derived by substituting the process output y in the process model (20.6) by r_y and then solving for y, giving

$$z_{\text{ff}} = \dot{r}_{y_f} \tag{20.15}$$

where index f indicates lowpass filter which may be of first order. A pure time differentiation should not be implemented because of noise amplification by the differentiation. Therefore the reference should be lowpass filtered before its time derivative is calculated.

The control function for the process (20.6) based on the sum of the feedback control function and the feedforward control function is as follows:[2]

$$\begin{aligned} z &= z_{\text{fb}} + z_{\text{ff}} && (20.16) \\ &= \underbrace{K_p e + K_i \int_0^t e d\tau}_{z_{\text{fb}}} + \underbrace{\dot{r}_{y_f}}_{z_{\text{ff}}} && (20.17) \end{aligned}$$

Now it is time to get the final control function, that is, the formula for the control vector u. From (20.5) we get

$$u = B^{-1}(z - f) \tag{20.18}$$

Here we use (20.17) to get the final control function:

$$\underline{u = B^{-1}\left(K_p e + K_i \int_0^t e d\tau + \dot{r}_{y_f} - f\right)} \tag{20.19}$$

If the reference is constant, as is the typical case in process control, the $\dot{r}_{y_f}$ term has value zero, and it can therefore be left out in the control function.

Figure 20.1 shows a block diagram of the control system.

Here are some characteristics of the control system:

- The controller is *model based* since it contains f and B from the process model.

[2] It is the sum because the process (the integrator) has a linear mathematical model.

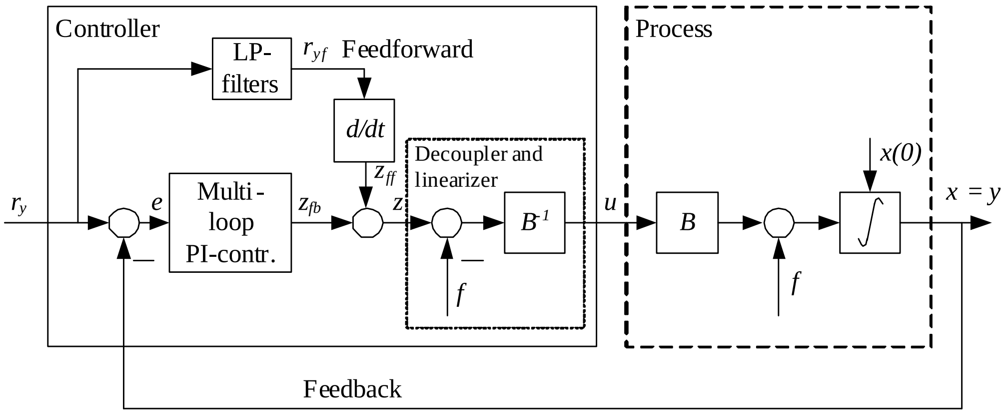

Figure 20.1: Block diagram of the control system based on feedback linearization

- Since the process disturbance is an argument of f and/or B the controller implements *feedforward* from the disturbance. (It also implements feedforward from the reference, due to the term $\dot{r}_{y_f}M$ in the controller.)
- The control system is *linear* even if the process is nonlinear.
- The control system consists of n *decoupled single-loop* control systems. This is illustrated in Figure 20.2.

Example 20.1 ***Feedback Linearization applied to level control***

In this example Feedback Linearization will be applied to a level control system. Figure 20.3 shows the control system. It assumed that the outflow is proportional to the control signal u and to the square root of the pressure drop along the control valve. The process model based on mass balance is (ρ is density)

$$\rho A\dot{h} = \rho q_{in} - \rho K_v u\sqrt{dP} \tag{20.20}$$

or

$$\dot{h} = \underbrace{\frac{q_{in}}{A}}_{=f} + \underbrace{\left(-\frac{K_v\sqrt{dP}}{A}\right)}_{=B} u \tag{20.21}$$

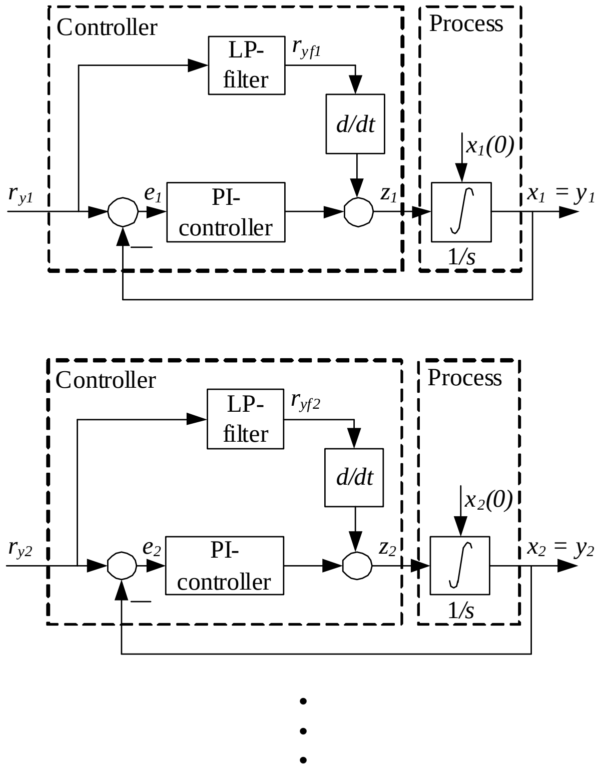

Figure 20.2: The control system consists of n decoupled single-loop control systems.

The control function becomes, cf. (20.19),

$$
\begin{aligned}
u &= B^{-1}\left(K_p e + K_i \int_0^t e d\tau + \dot{r}_{y_f} - f\right) &(20.22)\\
&= \left(-\frac{K_v\sqrt{dP}}{A}\right)^{-1}\left(K_p e + K_i \int_0^t e d\tau + \dot{r}_{y_f} - \frac{q_{in}}{A}\right) &(20.23)\\
&= -\frac{A}{K_v\sqrt{dP}}\left(K_p e + K_i \int_0^t e d\tau + \dot{r}_{y_f} - \frac{q_{in}}{A}\right) &(20.24)
\end{aligned}
$$

This control function requires that the differential pressure dP is measured, and that the inflow q_{in} is measured.

[End of Example 20.1]

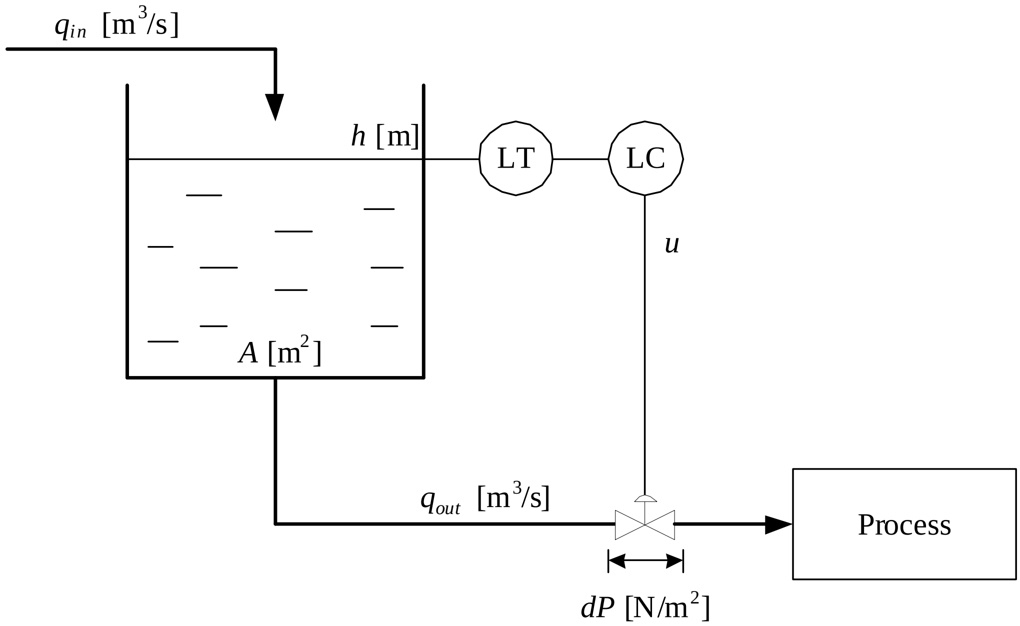

Figure 20.3: Example 20.1: Level control system

20.2.2 Case 2: Not all state variables are controlled

In Section 20.2.1 it is assumed that all the state variables are to be controlled, i.e. there is a reference for each of the state variables. Feedback linearization can be used also in cases where *not all* the state variables are controlled. The typical case is in positional control. Positions are only one set of the state variables. The other set is the velocities (rate of change of position). We will here focus on this typical case.

With position and velocity as state variables, the model of the process (e.g. motor or vessel) can be written on the following form where x is position, and u is the input (control variable), and y is the output:

$$\ddot{x} = f(x, \dot{x}, v) + B(x, \dot{x}, v) \cdot u \tag{20.25}$$

or simply

$$\ddot{x} = f + Bu \tag{20.26}$$

The output variable is the position:

$$y = x \tag{20.27}$$

By taking the second order time-derivative of (20.27) and using (20.26) we