Chapter 2

Frequency response

2.1 Introduction

The *frequency response* of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals – at least periodical signals – which excite systems, as the reference (setpoint) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of *frequency components*. Each frequency component is a sinusoidal signal having a certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system.

The frequency response is an important tool for analysis and design of signal filters (as lowpass filters and highpass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this chapter.

The definition of the frequency response – which will be given in the next section – *applies only to linear models*, but this linear model may very well be the local linear model about some operating point of a non-linear model.

The frequency response can found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.

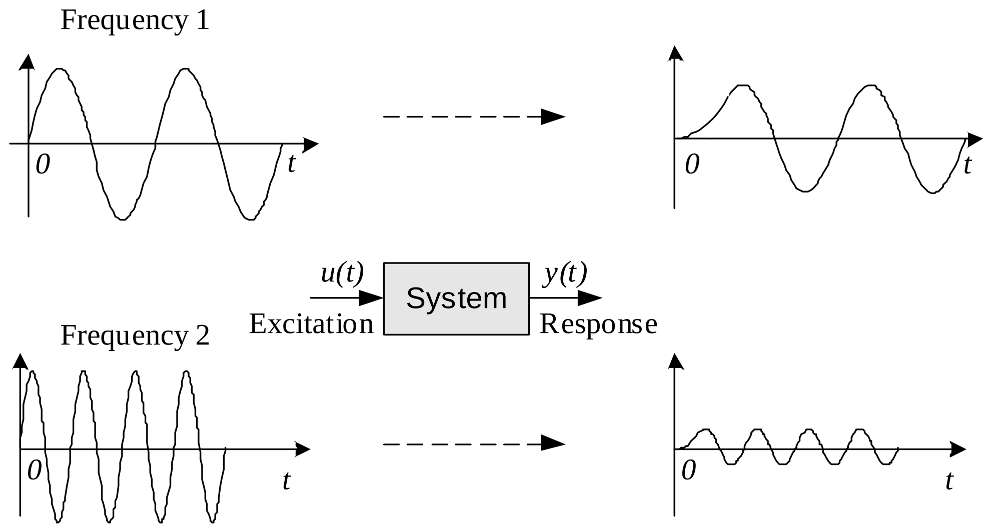

Figure 2.1: Sinusoidal signals in the input and the resulting responses on the output for two different frequencies

2.2 How to calculate frequency response from sinusoidal input and output

We can find the frequency response of a system by exciting the system with a sinusoidal signal of amplitude A and frequency ω [rad/s] and observing the response in the output variable of the system. [1] Mathematically, we set the input signal to

$$u(t) = U \sin \omega t \tag{2.1}$$

See Figure 2.1. This input signal will give a transient response (which will die, eventually) and a *steady-state* response, $y_s(t)$, in the output variable:

$$\begin{aligned} y_s(t) &= Y \sin(\omega t + \phi) && (2.2)\\ &= \underbrace{UA}_{Y} \sin(\omega t + \phi) && (2.3) \end{aligned}$$

Here A is the *(amplitude)gain,* and ϕ (phi) is *the phase lag* in radians. The frequency of $y_s(t)$ will be the same as in $u(t)$. Figure 2.2 shows in detail $u(t)$ and $y(t)$ for a simulated system. The system which is simulated is

$$y(s) = \frac{1}{s+1} u(s) \tag{2.4}$$

[1] The correspondance between a given frequency ω in rad/s and the same same frequency f in Hz is $\omega = 2\pi f$.

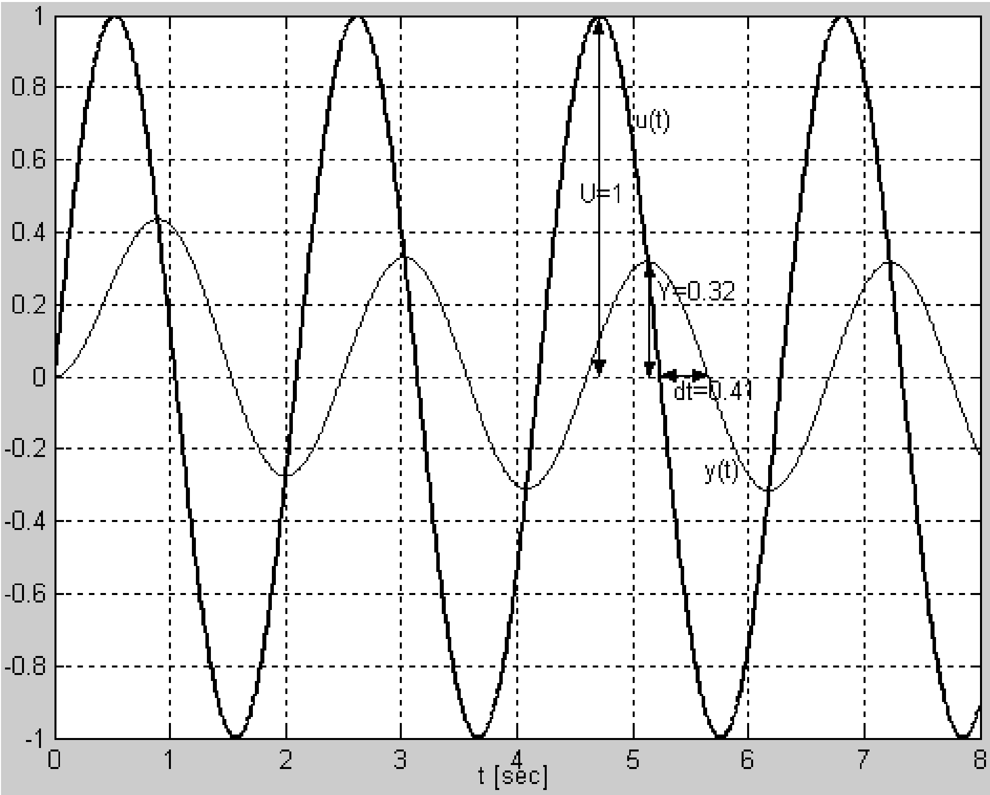

Figure 2.2: The input signal $u(t)$ and the resulting (sinusoidal) response $y(t)$ for a simulated system. $u(t)$ has frequency $\omega = 3$ rad/s and amplitude $U = 1$. The system is given by (2.4).

(a first order system with gain 1 and time-constant 1). The input signal $u(t)$ has frequency $\omega = 3$ rad/s and amplitude $U = 1$.

A is the ratio between the amplitudes of the output signal and the input signal (in steady-state):

$$A = \frac{Y}{U} \tag{2.5}$$

For the signals shown in Figure 2.2,

$$A = \frac{Y}{U} = \frac{0.32}{1} = 0.32 \tag{2.6}$$

ϕ can be calculated by first measuring the time-lag Δt between $u(t)$ and $y_s(t)$ and then calculating ϕ as follows:

$$\phi = -\omega \Delta t \quad \text{[rad]} \tag{2.7}$$

In Figure 2.2 we find $\Delta t = 0.41$ sec, which gives

$$\phi = -\omega \Delta t = -3 \cdot 0.41 = -1.23 \text{ rad} \tag{2.8}$$

The gain A and the phase-lag ϕ are functions of the frequency. We can use the following terminology: $A(\omega)$ is the *gain function*, and $\phi(\omega)$ is the *phase shift function* (or more simply: phase function). We say that $A(\omega)$ and $\phi(\omega)$ expresses the *frequency response* of the system.

Bode diagram

It is common to present $A(\omega)$ and $\phi(\omega)$ graphically in a *Bode diagram*, which consists of two subdiagrams, one for $A(\omega)$ and one for $\phi(\omega)$, where the phase values are usually plotted in degrees (not radians). Figure 2.3 shows a Bode diagram of the frequency response of the system given by (2.4). The curves may stem from a number of A-values and ϕ-values found in experiments (or simulations) with an sinusoidal input signal of various frequencies. The curves may also stem from the transfer function of the system, as described in Section 2.3. The frequency axes usually show the 10-logarithm of the frequency in rad/s or in Hz.

Actually, the system (2.4) is used to generate $u(t)$ and $y(t)$ shown in Figure 2.2. We have earlier in this chapter calculated $A(3) = 0.32 = -10.2$ dB (the dB-unit is described below) and phase lag $\phi(3) = -1.23$ rad $=$ -72 degrees. This gain value and phase lag value are indicated in the Bode diagram in Figure 2.3.

The $A(\omega)$-axis is usually drawn with decibel (dB) as unit. The decibel value of a number x is calculated as

$$x \text{ [dB]} = 20 \log_{10} x \tag{2.9}$$

Table 2.1 shows some examples of dB-values.

2.3 How to calculate frequency response from transfer functions

In Section 2.2 we saw how to find the frequency response from experiments on the system. No model was assumed. However, if we know a transfer function model of the system, we can *calculate the frequency response from the transfer function*, as explained below.

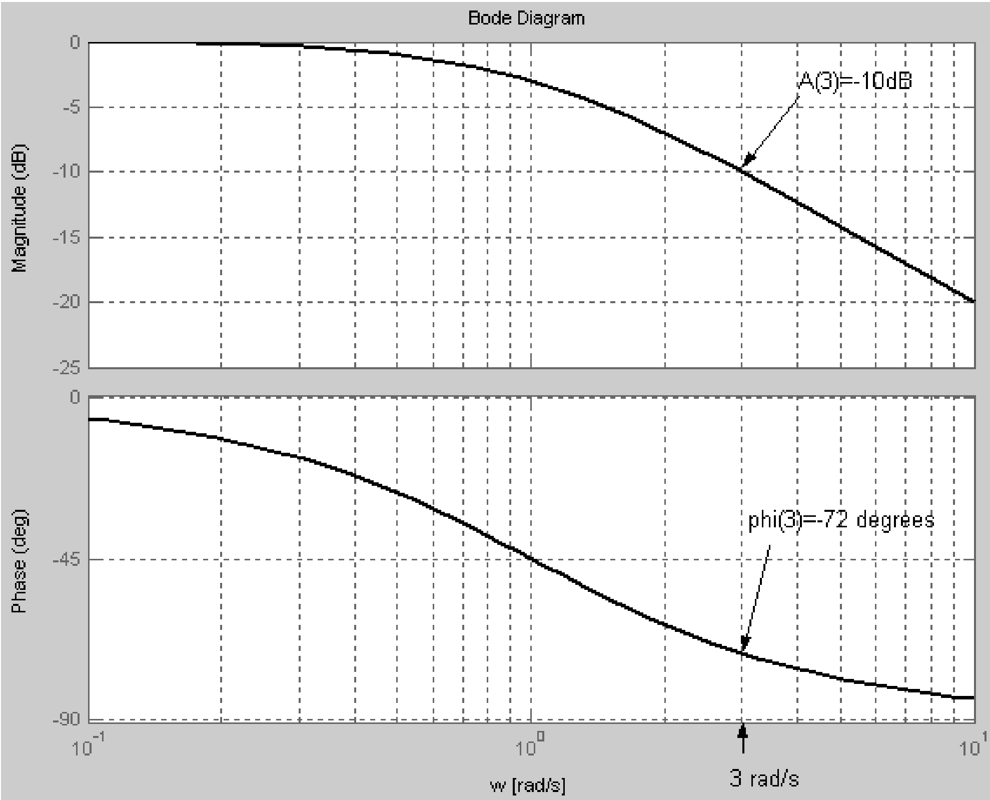

Figure 2.3: The frequency response of the system given by (2.4) presented in a Bode diagram

Suppose that system has the transfer function $H(s)$ from input u to output y, that is,

$$y(s) = H(s)u(s) \tag{2.10}$$

By setting

$$s = j\omega \tag{2.11}$$

(j is the imaginary unit) into $H(s)$, we get the complex quantity $H(j\omega)$, which is the *frequency response* (function). The *gain function* is

$$A(\omega) = |H(j\omega)| \tag{2.12}$$

and the *phase shift function* is the angle or argument of $H(j\omega)$:

$$\phi(\omega) = \arg H(j\omega) \tag{2.13}$$

(The formulas (2.12) and (2.13) can be derived using the Laplace transform.)