Chapter 13

Stability analysis of discrete-time dynamic systems

13.1 Definition of stability properties

Assume given a dynamic system with input u and output y. The stability property of a dynamic system can be defined from the *impulse response*[1] of a system as follows:

- **Asymptotic stable system:** The steady state impulse response is zero:
$$\lim_{k\to\infty} y_\delta(k) = 0 \tag{13.1}$$
- **Marginally stable system:** The steady state impulse response is different from zero, but limited:
$$0 < \lim_{k\to\infty} y_\delta(k) < \infty \tag{13.2}$$
- **Unstable system:** The steady state impulse response is unlimited:
$$\lim_{k\to\infty} y_\delta(k) = \infty \tag{13.3}$$

The impulse response for the different stability properties are illustrated in Figure 13.1. (The simulated system is defined in Example 13.1.)

[1] An impulse $\delta(0)$ is applied at the input.

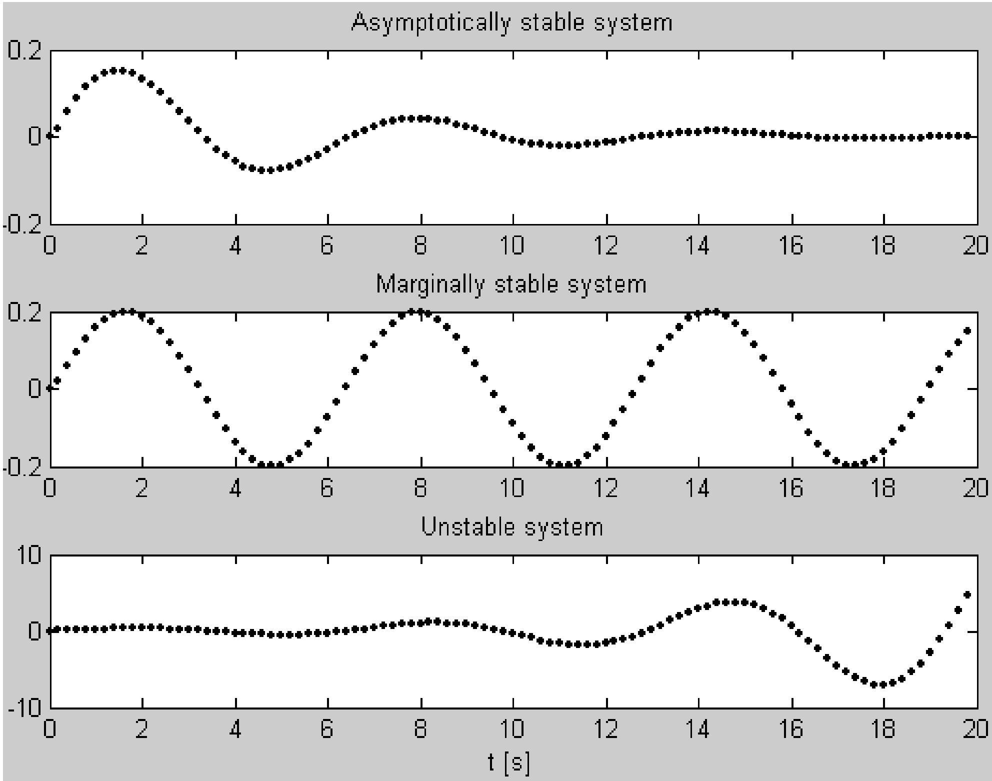

Figure 13.1: Impulse response and stability properties

13.2 Stability analysis of transfer function models

In the following we will base the analysis on the following fact: *The transfer function is the z-transformed impulse response.* Here is the proof of this fact: Given a system with transfer function $H(z)$. Assume that the input u is an impulse, which is a signal having value 1 at time index $k = 0$ and value zero at other points of time. According to (10.7) $u(z) = 1$. Then the z-transformed impulse response is

$$y(z) = H(z)u(z) = H(z) \cdot 1 = H(z) \tag{13.4}$$

(as stated).

Now, we proceed with the stability analysis of transfer functions. The impulse response $y_\delta(k)$, which defines the stability property of the system, is determined by the poles of the system's poles and zeros since the impulse responses is the inverse z-transform of the transfer function:

$$y_\delta(k) = \mathcal{Z}^{-1}\{H(z)\} \tag{13.5}$$

Consequently, the stability property is determined by the poles and zeros of $H(z)$. However, we will soon see that only the poles determine the stability.

We will now derive the relation between the stability and the poles by studying the impulse response of the following system:

$$H(z) = \frac{y(z)}{u(z)} = \frac{bz}{z-p} \tag{13.6}$$

The pole is p. Do you think that this system is too simple as a basis for deriving general conditions for stability analysis? Actually, it is sufficient because we can always think that a given z-transfer function can be partial fractionated in a sum of partial transfer functions or terms each having one pole. Using the superposition principle we can conclude about the stability of the original transfer function.

In the following, cases having of multiple (coinciding) poles will be discussed, but the results regarding stability analysis will be given.

The system given by (13.6) has the following impulse response calculated below. It is assumed that the pole in general is a complex number which may be written on polar form as

$$p = me^{j\theta} \tag{13.7}$$

where m is the magnitude and θ the phase. The impulse response is

$$y_\delta(k) = \mathcal{Z}^{-1}\left\{\frac{bz}{z-p}\right\} \tag{13.8}$$

$$= \mathcal{Z}^{-1}\left\{\frac{p}{1-pz^{-1}}\right\} \tag{13.9}$$

$$= \mathcal{Z}^{-1}\left\{b\sum_{k=0}^{\infty} p^k z^{-k}\right\} \tag{13.10}$$

$$= bp^k \tag{13.11}$$

$$= b|m|^k e^{jk\theta} \tag{13.12}$$

From (13.12) we see that it is the *magnitude* m which determines if the steady state impulse response converges towards zero or not. From (13.12) we can now state the following relations between stability and pole placement (the statements about multiple poles have however not been derived here):

- **Asymptotic stable system:** All poles lie inside (none is on) the unit circle, or what is the same: all poles have magnitude less than 1.

- **Marginally stable system:** One or more poles – but no multiple poles – are on the unit circle.
- **Unstable system:** At least one pole is outside the unit circle. Or: There are multiple poles on the unit circle.

The "stability areas" in the complex plane are shown in Figure 13.2.

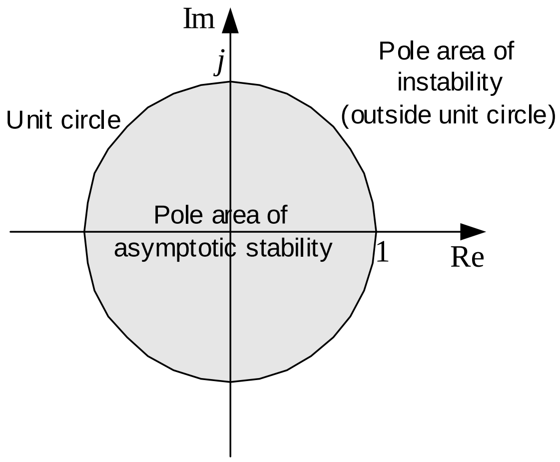

Figure 13.2: The different stability property areas of the complex plane

Let us return to the question about the relation between the *zeros* and the stability. We consider the following system:

$$H_1(z) = \frac{y(z)}{u(z)} = \frac{b(z-c)}{z-p} = (z-c)H(z) \tag{13.13}$$

where $H(z)$ is it the "original" system (without zero) which were analyzed above. The zero is c. $H_1(z)$ can be written as

$$H_1(z) = \frac{bz}{z-p} + \frac{-bc}{z-p} \tag{13.14}$$

$$= H(z) - cz^{-1}H(z) \tag{13.15}$$

The impulse response of $H_1(z)$ becomes

$$y_{\delta_1}(k) = y_\delta(k) - cy_\delta(k-1) \tag{13.16}$$

where $y_\delta(k)$ is the impulse response of $H(z)$. We see that the zero does not influence wether the steady state impulse response converges towards to zero or not. We draw the conclusion that the zeros of the transfer function do not influence the stability of the system.

Example 13.1 ***Stability analysis of discrete-time system***

The three responses shown in Figure 13.1 are actually the impulse responses in three systems each having a transfer function on the form

$$\frac{y(z)}{u(z)} = H(z) = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0} \tag{13.17}$$

The parameters of the systems are given below:

1. Asymptotically stable system: $b_1 = 0.019$, $b_0 = 0.0190$, $a_1 = -1.885$ and $a_0 = 0.923$. The poles are

   $$z_{1,\,2} = 0.94 \pm j0.19 \tag{13.18}$$

   They are shown in Figure 13.3 (the zero is indicated by a circle). The poles are inside the unity circle.

2. Marginally stable system: $b_1 = 0.020$, $b_0 = 0.020$, $a_1 = -1.96$ and $a_0 = 1.00$. The poles are

   $$z_{1,\,2} = 0.98 \pm j0.20 \tag{13.19}$$

   They are shown in Figure 13.3. The poles are on the unity circle.

3. Unstable system: $b_1 = 0.021$, $b_0 = 0.021$, $a_1 = -2.04$ and $a_0 = 1.08$. The poles are

   $$z_{1,\,2} = 1.21 \pm j0.20 \tag{13.20}$$

   They are shown in Figure 13.3. The poles are outside the unity circle.

[End of Example 13.1]

13.3 Stability analysis of state space models

Assume that the system has the following state space model:

$$x(k+1) = Ax(k) + Bu(k) \tag{13.21}$$

$$y(k) = Cx(k) + Du(k) \tag{13.22}$$

We can determine the stability by finding the corresponding transfer function from u to y, and then calculating the poles from the transfer function, as we did in the previous section. Let's derive the transfer

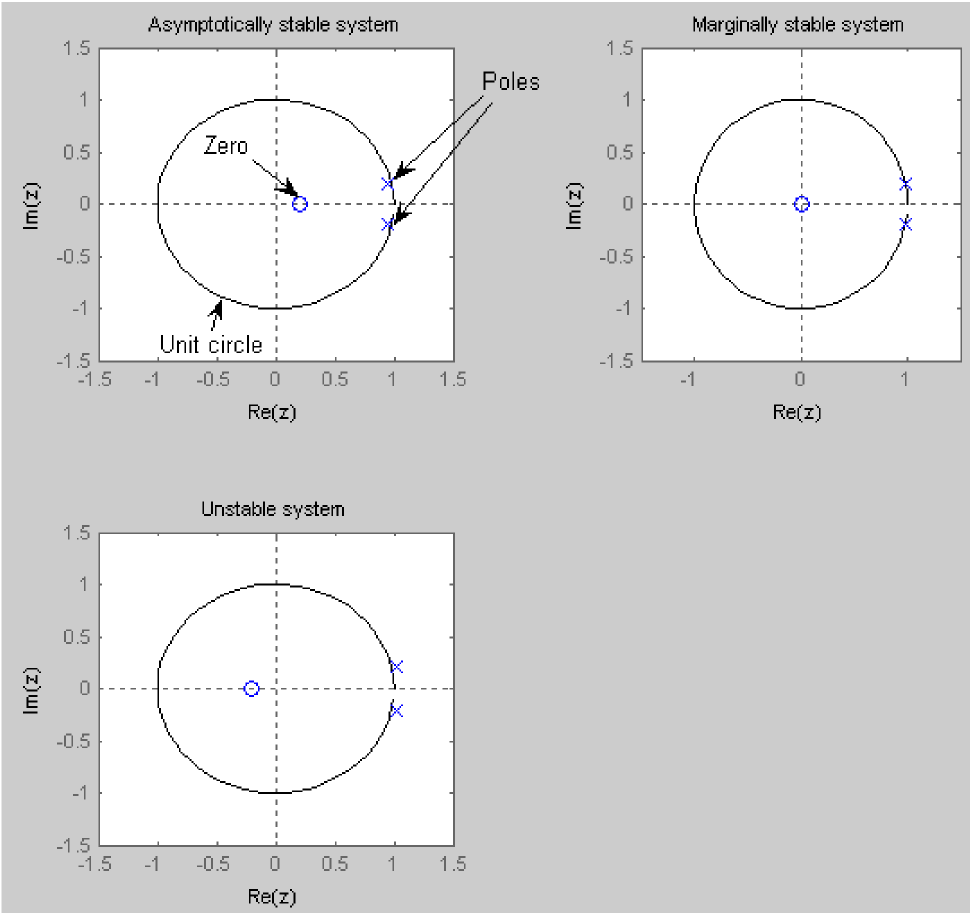

Figure 13.3: Example 13.1: Poles (and zeros) for the three systems each having different stability property

function: Take the $\mathcal{Z}$-transform of (13.21) – (13.22) to get (I is the identity matrix of equal dimension as of A)

$$zIx(z) = Ax(z) + Bu(z) \tag{13.23}$$

$$y(z) = Cx(z) + Du(z) \tag{13.24}$$

Solving (13.23) for $x(z)$ gives

$$x(z) = (zI - A)^{-1}Bu(z) \tag{13.25}$$

Inserting this $x(z)$ into (13.24) gives

$$y(z) = \left[C(zI - A)^{-1}B + D\right]u(z) \tag{13.26}$$