

5.4 Stability margins

5.4.1 Stability margins in terms of gain margin and phase margin

An asymptotically stable feedback system may become marginally stable if the loop transfer function changes. The *gain margin* GM and the *phase margin* PM [radians or degrees] are *stability margins* which in their own ways express how large parameter changes can be tolerated before an asymptotically stable system becomes marginally stable. Figure 5.7 shows the stability margins defined in the Nyquist diagram. GM is the

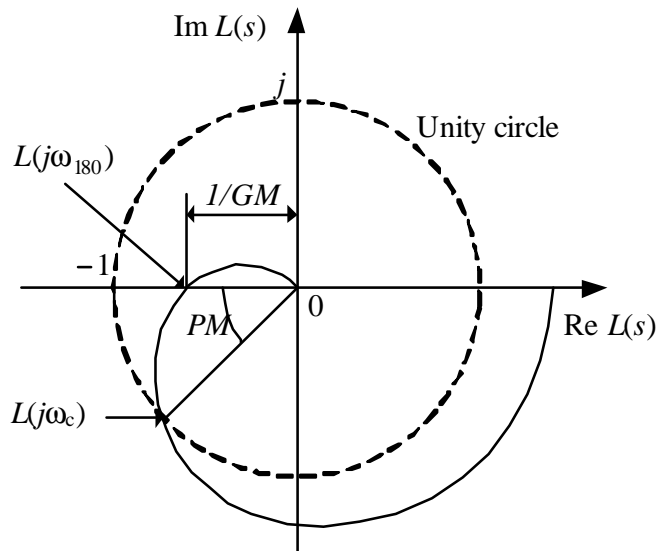


Figure 5.7: Gain margin GM and phase margin PM defined in the Nyquist diagram

(multiplicative, not additive) increase of the gain that L can tolerate at ω_{180} before the L curve (in the Nyquist diagram) passes through the critical point. Thus,

$$|L(j\omega_{180})| \cdot GM = 1 \quad (5.28)$$

which gives

$$GM = \frac{1}{|L(j\omega_{180})|} = \frac{1}{|\operatorname{Re} L(j\omega_{180})|} \quad (5.29)$$

(The latter expression in (5.29) is because at ω_{180} , $\operatorname{Im} L = 0$ so that the amplitude is equal to the absolute value of the real part.)

If we use decibel as the unit (like in the Bode diagram which we will soon encounter), then

$$GM \text{ [dB]} = -|L(j\omega_{180})| \text{ [dB]} \quad (5.30)$$

The phase margin PM is the phase reduction that the L curve can tolerate at ω_c before the L curve passes through the critical point. Thus,

$$\arg L(j\omega_c) - PM = -180^\circ \quad (5.31)$$

which gives

$$PM = 180^\circ + \arg L(j\omega_c) \quad (5.32)$$

We can now state as follows: The feedback (closed) system is asymptotically stable if

$$GM > 0\text{dB} = 1 \text{ and } PM > 0^\circ \quad (5.33)$$

This criterion is often denoted the *Bode-Nyquist stability criterion*.

Reasonable ranges of the stability margins are

$$2 \approx 6\text{dB} \leq GM \leq 4 \approx 12\text{dB} \quad (5.34)$$

and

$$30^\circ \leq PM \leq 60^\circ \quad (5.35)$$

The larger values, the better stability, but at the same time the system becomes more sluggish, dynamically. If you are to use the stability margins as design criterias, you can use the following values (unless you have reasons for specifying other values):

$$GM \geq 2.5 \approx 8\text{dB} \text{ and } PM \geq 45^\circ \quad (5.36)$$

For example, the controller gain, K_p , can be adjusted until one of the inequalities becomes an equality.³

It can be shown⁴ that for $PM \leq 70^\circ$, the damping of the feedback system approximately corresponds to that of a second order system with relative damping factor

$$\zeta \approx \frac{PM}{100^\circ} \quad (5.37)$$

For example, $PM = 50^\circ \sim \zeta = 0.5$.

³But you should definitely check the behaviour of the control system by simulation, if possible.

⁴The result is based on the assumption that the loop transfer function is $L(s) = \omega_0^2 / [(s + 2\zeta\omega_0)s]$ which gives tracking transfer function $T(s) = L(s) / [1 + L(s)] = \omega_0^2 / [s^2 + 2\zeta\omega_0 s + \omega_0^2]$. The phase margin PM can be calculated from $L(s)$.

5.4.2 Stability margins in terms of maximum sensitivity amplitude

An alternative quantity of a stability margin is the minimum distance from the $L(j\omega)$ curve to the critical point. This distance is $|1 + L(j\omega)|$, see Figure 5.8. So, we can use the minimal value of $|1 + L(j\omega)|$ as a stability

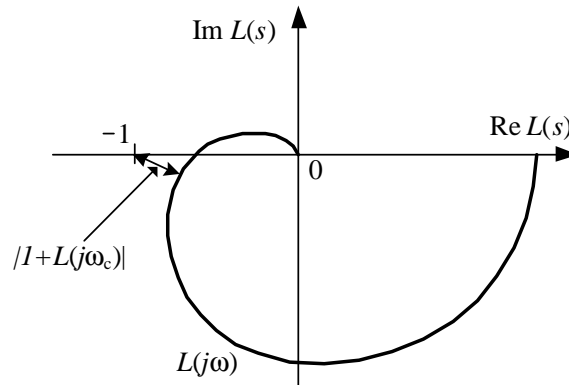


Figure 5.8: The distance between the $L(j\omega)$ curve and the critical point is $|1 + L|$. The minimum of this distance is related to the stability margin.

margin. However, it is more common to take the inverse of the distance: Thus, a stability margin is the *maximum* value of $1/|1 + L(j\omega)|$. And since $1/[1 + L(s)]$ is the sensitivity function $S(s)$, then $|S(j\omega)|_{\max}$ represents a stability margin. Reasonable values are in the range

$$1.5 \approx 3.5 \text{ dB} \leq |S(j\omega)|_{\max} \leq 3.0 \approx 9.5 \text{ dB} \quad (5.38)$$

If you use $|S(j\omega)|_{\max}$ as a criterion for adjusting controller parameters, you can use the following criterion (unless you have reasons for some other specification):

$$|S(j\omega)|_{\max} = 2.0 \approx 6 \text{ dB} \quad (5.39)$$

5.5 Stability analysis in a Bode diagram

It is most common to use a Bode diagram for frequency response based stability analysis of closed loop systems. The Nyquist's Stability Criterion says: The closed loop system is marginally stable if the Nyquist curve (of L) goes through the critical point, which is the point $(-1, 0)$. But where is the critical point in the Bode diagram? The critical point has phase (angle) -180° and amplitude $1 = 0\text{dB}$. The critical point therefore

constitutes two lines in a Bode diagram: The 0dB line in the amplitude diagram and the -180° line in the phase diagram. Figure 5.9 shows typical L curves for an asymptotically stable closed loop system. In the figure, GM , PM , ω_c and ω_{180} are indicated.

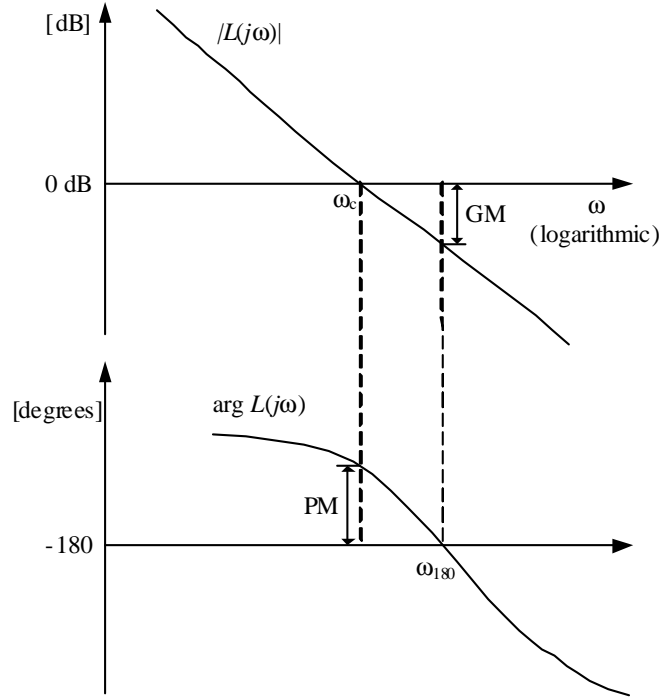


Figure 5.9: Typical L curves of an asymptotically stable closed loop system with GM , PM , ω_c and ω_{180} indicated

Example 5.2 Stability analysis of a feedback control system

Given a feedback control system with structure as shown in Figure 5.10. The loop transfer function is

$$L(s) = H_c(s)H_p(s) = \underbrace{K_p}_{H_c(s)} \underbrace{\frac{1}{(s+1)^2 s}}_{H_p(s)} = \frac{K_p}{(s+1)^2 s} = \frac{n_L(s)}{d_L(s)} \quad (5.40)$$

We will determine the stability property of the control system for different values of the controller gain K_p in three ways: Pole placement, Nyquist's Stability Criterion, and simulation. The tracking transfer function is

$$T(s) = \frac{y_m(s)}{y_{mSP}(s)} = \frac{L(s)}{1+L(s)} = \frac{n_L(s)}{d_L(s) + n_L(s)} = \frac{K_p}{s^3 + 2s^2 + s + K_p} \quad (5.41)$$

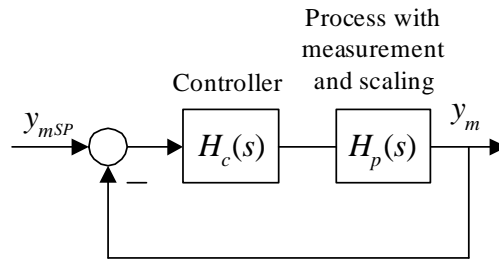


Figure 5.10: Example 5.2: Block diagram of feedback control system

The characteristic polynomial is

$$c(s) = s^3 + 2s^2 + s + K_p \quad (5.42)$$

Figures 5.11 – 5.13 show the step response after a step in the setpoint, the poles, the Bode diagram and Nyquist diagram for three K_p values which result in different stability properties. The detailed results are shown below.

- $K_p = 1$: Asymptotically stable system, see Figure 5.11. From the Bode diagram we read off stability margins $GM = 6.0\text{dB} = 2.0$ and $PM = 21^\circ$. we see also that $|S(j\omega)|_{\max} = 11\text{ dB} = 3.5$ (a large value, but it corresponds with the small the phase margin of $PM = 20^\circ$).
- $K_p = 2$: Marginally stable system, see Figure 5.12. From the Bode diagram, $\omega_c = \omega_{180}$. The L curve goes through the critical point in the Nyquist diagram. $|S|_{\max}$ has infinitely large value (since the minimum distance, $1/|S|_{\max}$, between $|L|$ and the critical point is zero).

Let us calculate the period T_p of the undamped oscillations: Since $\omega_{180} = 1.0\text{rad/s}$, the period is $T_p = 2\pi/\omega_{180} = 6.28\text{s}$, which fits well with the simulation shown in Figure 5.12.

- $K_p = 4$: Unstable system, see Figure 5.13. From the Bode diagram, $\omega_c > \omega_{180}$. From the Nyquist diagram we see that the L curve passes outside the critical point. (The frequency response curves of M and N have no physical meaning in this the case.)

[End of Example 5.2]

5.6 Robustness in term of stability margins

Per definition the stability margins expresses the robustness of the feedback control system against certain parameter changes in the loop transfer function:

- *The gain margin GM* is how much the loop gain, K , can increase before the system becomes unstable. For example, is $GM = 2$ when $K = 1.5$, the control system becomes unstable for K larger than $1.5 \cdot 2 = 3.0$.
- *The phase margin PM* is how much the phase lag function of the loop can be reduced before the loop becomes unstable. One reason of reduced phase is that the time delay in control loop is increased. A change of the time delay by $\Delta\tau$ introduces the factor $e^{-\Delta\tau s}$ in $L(s)$ and contributes to $\arg L$ with $-\Delta\tau \cdot \omega$ [rad] or $-\Delta\tau \cdot \omega \frac{180^\circ}{\pi}$ [deg]. $|L|$ is however not influenced because the amplitude function of $e^{-\tau s}$ is 1, independent of the value of τ . The system becomes unstable if the time delay have increased by $\Delta\tau_{\max}$ such that⁵

$$PM = \Delta\tau_{\max} \cdot \omega_c \frac{180^\circ}{\pi} \text{ [deg]} \quad (5.43)$$

which gives the following maximum change of the time delay:

$$\Delta\tau_{\max} = \frac{PM}{\omega_c} \frac{\pi}{180^\circ} \quad (5.44)$$

If you want to calculate how much the phase margin PM is reduced if the time delay is increased by $\Delta\tau$, you can use the following formula which stems from (5.43):

$$\Delta PM = \Delta\tau \cdot \omega_c \frac{180^\circ}{\pi} \text{ [deg]} \quad (5.45)$$

For example, assume that a given control system has $\omega_c = 0.2\text{rad/min}$ and $PM = 50^\circ$. If the time delay increases by 1min, the phase margin is reduced by $\Delta PM = 1 \cdot 0.2 \frac{180^\circ}{\pi} = 11.4^\circ$, i.e. from 50° to 38.6° .

⁵Remember that PM is found at ω_c .

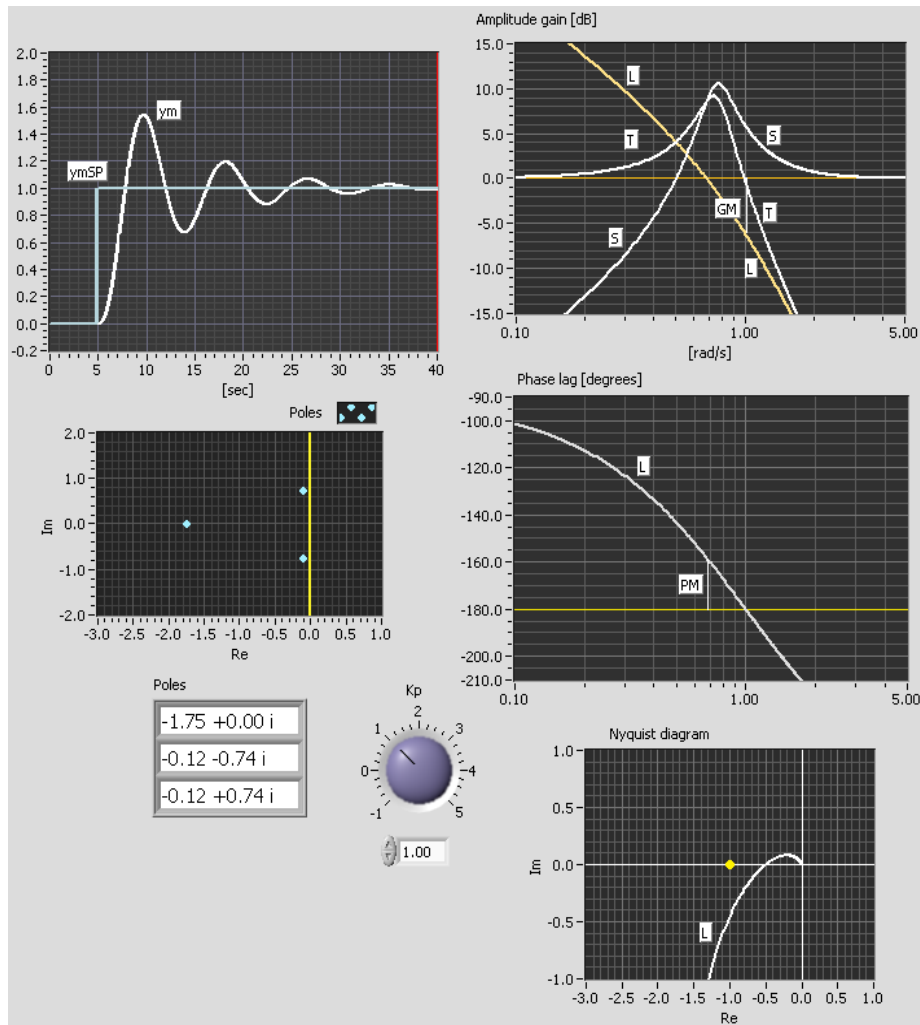


Figure 5.11: Example 5.2: Step response (step in setpoint), poles, Bode diagram and Nyquist diagram with $K_p = 1$. The control system is asymptotically stable.