- **Unbiased estimate:**
$$S = \frac{1}{N-L} \tag{15.20}$$
This calculates $R_{xy}(L)$ as an average value of the product $[x(k+L)-m_x][y(k)-m_y]$. However, $R_{xy}(L)$ may be very "noisy" if L is large since then the summation is calculated with only a few additive terms (it is assumed that there are noise or random components in x and/or y).

- **Biased estimate:**
$$S = \frac{1}{N} \tag{15.21}$$
With this option $R_{xy}(L)$ is not an average value of the product $[x(k+L)-m_x][y(k)-m_y]$ since the sum of terms is divided by N no matter how many additive terms there are in the the summation. Although this makes $R_{xy}(L)$ become "biased" it reduces the "noise" in $R_{xy}(L)$ because the "noisy" terms are weighed by $1/N$ in stead of $1/(N-L)$. Unless you have reasons for some other selection, you may use biased estimate as the default option.

Correlation (auto/cross) is the same as covariance (auto/cross) except that the mean value, as m_x, is removed from the formulas. Hence, the cross-correlation is, cf. (15.15),

$$r_{xy}(L) = E\{x(k+L)y(k)\} \tag{15.22}$$

15.3 White and coloured noise

15.3.1 White noise

An important type of stochastic signals are the so-called *white noise signals (or processes).* "White" is because in some sense white noise contains equally much of all frequency components, analogously to white light which contains all colours. White noise has zero mean value:

Mean value of white noise:
$$m_x = 0 \tag{15.23}$$

There is no co-variance or relation between sample values at different time-indexes, and hence the auto-covariance is zero for all lags L except for $L = 0$. Thus, the auto-covariance is the pulse function shown in Figure 15.4. Mathematically the auto-covariance function of white noise is

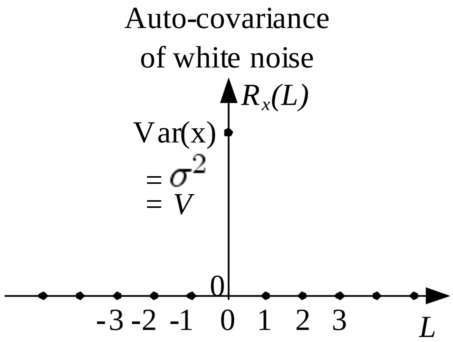

Figure 15.4: White noise has auto-correlation function like a pulse function.

$$R_x(L) = \mathrm{Var}(x)\delta(L) = \sigma^2\delta(L) = V\delta(L) \tag{15.24}$$

Here, the short-hand symbol V has been introduced for the variance. $\delta(L)$ is the *unit pulse* defined as follows:

Unit pulse:

$$\delta(L) = \begin{cases} 1 \text{ when } L = 0 \\ 0 \text{ when } L \neq 0 \end{cases} \tag{15.25}$$

White noise is an important signal in estimation theory because the random noise which is aways present in measurements, can be represented by white noise. For example, the variance of the assumed white measurement noise is used as an input parameter in the Kalman Filter design, cf. Chapter 18.

If you calculate the auto-covariance of a white noise sequence of finite length, the auto-covariance function will not be excactly as the ideal function shown in Figure 15.4, but the main characteristic showing a relatively large value at lag $L = 0$ is there.

Example 15.1 ***White noise***

Figure 15.5 shows a simulated white noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x.[1] The white noise characteristic of the signal is clearly indicated by $R_x(L)$.

[End of Example 15.1]

[1] Implemented in LabVIEW.

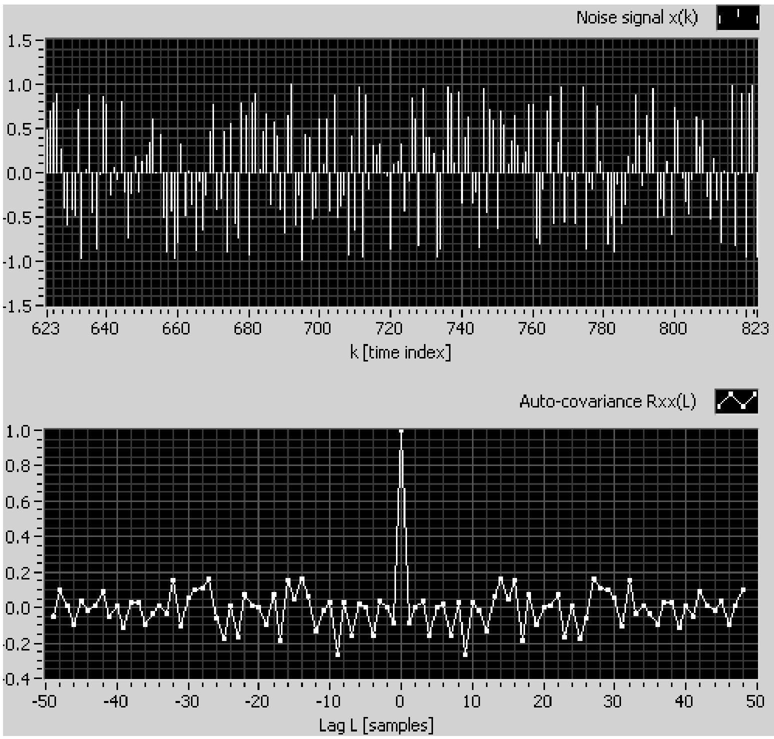

Figure 15.5: Example 15.1: Simulated white noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x.

15.3.2 Coloured noise

As opposite to white noise, *coloured noise* does not vary completely randomly. In other words, there is a co-variance between the sample values at different time-indexes. As a consequence, the auto-covariance $R_x(L)$ is non-zero for lags $L \neq 0$. $R_x(L)$ will have a maximum value at $L = 0$, and $R_x(L)$ will descrease for increasing L.

You may generate coloured noise from white noise by sending the white noise through a dynamic system, typically a lowpass filter. Such a system is denoted *shaping filter*. The output signal of the shaping filter will be coloured noise. You can tune the colour of the coloured noise by adjusting the parameters of the shaping filter.

Example 15.2 ***Coloured noise***

Figure 15.6 shows a simulated coloured noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x.[2] The coloured noise is the output of this shaping filter:

$$x(k) = ax(k-1) + (1-a)v(k) \tag{15.26}$$

which is a discrete-time first order lowpass filter. The filter input $v(k)$ is white noise. The filter parameter is $a = 0.98$. (If the filter parameter is 0 the filter performs no filtering, and the output is just white noise.) The

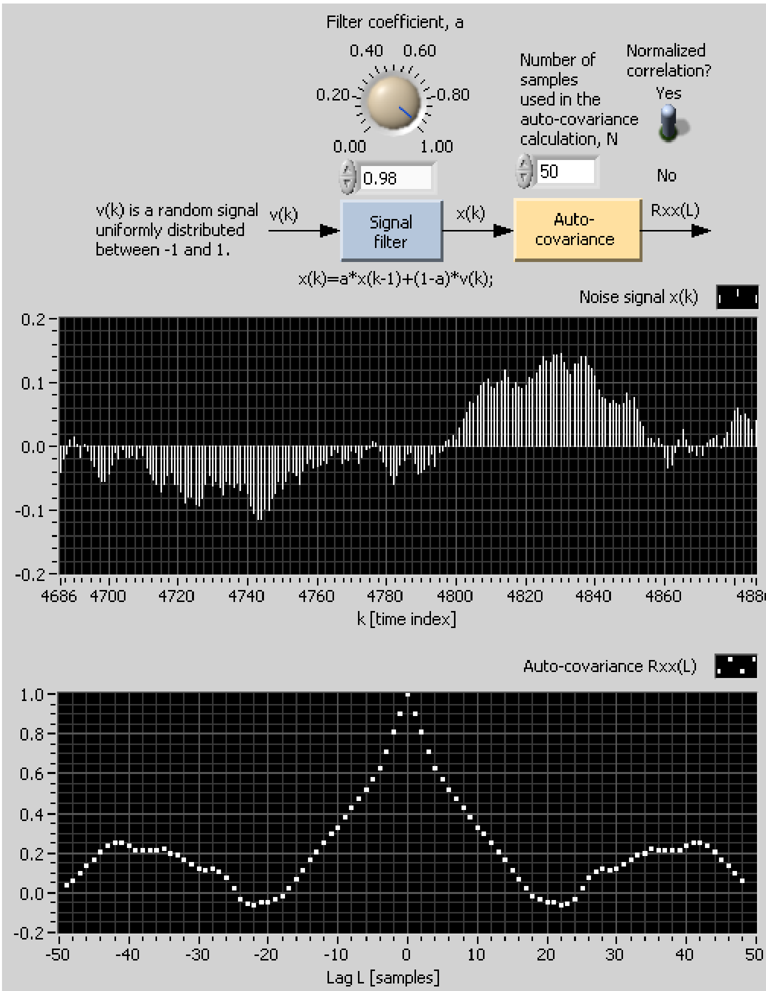

Figure 15.6: Example 15.1: Simulated coloured noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x.

coloured noise characteristic of the signal is shown both in the plot of the

[2] Implemented in LabVIEW.

signal $x(k)$ in the upper diagram of Figure 15.6 and in the auto-covariance $R_x(L)$ shown in the lower diagram of Figure 15.6.

[End of Example 15.2]

15.4 Propagation of mean value and co-variance through static systems

If a stochastic ("random") signals excites a static or dynamic system, the mean value and the co-variance of the output signal is different from those of the input. In this section we will concentrate on *static* systems. The results are useful e.g. in calculating the system gain needed to obtain a random signal of a specified variannce when the source signal is a random signal of fixed variance.

The theory of the propagation of mean value and co-variance through *dynamic* systems is certainly important if you are going to analyze and design signal filters, controllers and state estimators assuming they are excited by random signals. However, it is my experience that this theory is not needed to be able to use the tools that exist for such applications (e.g. the Kalman Filter for state estimation). Therefore, I have omitted the topic of propagation of mean value and co-variance through dynamic systems in this book.

Assume given the following static linear system:

$$y(k) = Gv(k) + C \tag{15.27}$$

where v is a stationary stochastic input signal with mean value m_v and co-variance $R_v(L)$. y is the output of the system. G is the gain of the system, and C is a constant. In a multivariable system G is a matrix and C is a vector, but in the following we will assume that G and C are scalars, which is the most usual case.

Let us calculate the mean value and the auto-covariance of the output y. The mean value becomes

$$\begin{aligned} m_y &= E[y(k)] = E[] = GE\left[v(k)\right] + C && (15.28) \\ &= Gm_v + C && (15.29) \end{aligned}$$