Chapter 1

Introduction

1.1 Dynamic Systems

This book gives an introduction to the systems theory for dynamic systems. Dynamic means “which has to do with the movement and change”. Dynamic systems are systems where the variables can vary or develop as functions of time. We say that dynamic systems have dynamic time-responses. A few examples of dynamic systems:

- A liquid tank (the level may vary as a function of time).
- A motor (the speed may vary).
- A heated tank (the temperature may vary).
- An electrical circuit (the output voltage may vary).
- A position control system for a robot manipulator (the position may vary).
- A signal filter (the filter output signal may vary).

Figure 1.1 gives an illustration. The figure shows a block diagram of a dynamic system. The input variable is a step function, and the time-response in the output variable has a dynamic time-response.

Physical systems are dynamic systems, control systems for such processes are dynamic systems, and signal filters are dynamic systems. Systems theory for dynamic systems is a key to describe, analyze and design such systems.

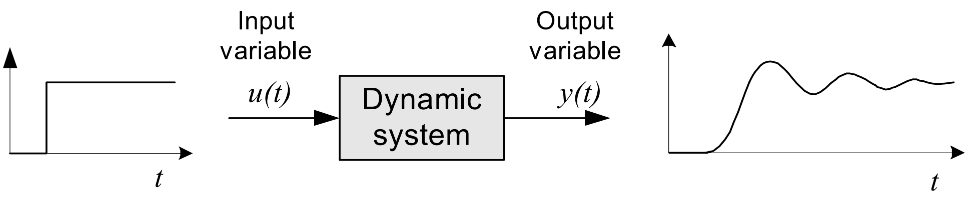

Figure 1.1: Dynamic systems are systems where the variables can vary or develop as functions of time.

1.2 The Contents of This Book

Here is a short description of the contents of the book:

- Chapter 2, **Mathematical Modeling**, describes how to develop mathematical models for dynamic systems and for static systems. We will see that the models for dynamic systems will be in the form of differential equations. There are examples of modeling mass systems, thermal systems, mechanical systems, and electrical systems. Simulated responses are shown. The chapter does not describe theoretical analysis of the models, but this is covered by later chapters.
- Chapter 3, **Model Forms and Time-Response Calculations**, describes two model forms, namely *differential equations* and *transfer functions*. . It is shown how differential equations can be presented on the standard form called state-space model which is just a set of first order differential equations. A transfer function is a compact model form which is found by taking the Laplace transform of the differential equation model.

  The chapter also shows how we can *calculate time-responses* both analytically and numerically.

  The chapter also describes *linearization* of non-linear models.
- Chapter 4, **Standard Transfer Functions and Dynamics**, shows how we can analyze *dynamic properties* via standard transfer function models. Such transfer functions represent integrator systems, first order systems, second order systems, and time-delay systems. The chapter defines parameters such as gain, time-constant, resonance frequency, and damping factor.
- Chapter 5, **Frequency Response**, defines the *frequency response* of

dynamic systems. The frequency response is a frequency dependent function which expresses the response in the output variable of a dynamic system due to a sinusoidal input variable. Frequency response is very important tool in the analysis and design of signal filters (as lowpass filters and highpass filters), but it is also were very useful for analysis and design of control systems. In this chapter the most important filtering functions are defined. The filtering analysis is useful not only for filter analysis, but also for the analysis of physical processes (for example a tank in a process string acts as a lowpass filter) and control systems (a control system behaves much like lowpass filter). Discrete-time filter functions are described briefly at the end of the chapter.

- Chapter 6, **Stability Analysis**, defines different stability properties for dynamic systems from their time-responses and shows how the stability property can be determined from the poles of the system or from the eigenvalues of the system. Stability analysis is particularly important in control theory since control systems can get poor stability or even become unstable if the parameters of the controller are given wrong values.

- Chapter 7, **Estimation of Model Parameters**, shows how the least squares method can be used to estimate or calculate the value of unknown parameters in a mathematical model developed from time-series of process measurements. Also, development of black-box dynamic models, which are models with non-physical parameters which represents the input-output dynamic relation, is described briefly.

Chapter 2

Mathematical Modeling

2.1 Introduction

A *mathematical model* of a system is the set of equations which describe the behavior of the system. Examples of dynamic systems which we will learn to model, are liquid tanks, heated systems, motion systems, and electrical systems.

From the mathematical model of a dynamic system we can calculate the time-response in the system variables for any instant of time. From the model we can develop simulators, to analyze the stability properties, and to design systems, as physical processes, control systems and signal filters. As you certainly realize, mathematical models can be very useful.

Unfortunately we can never make a completely precise model of a physical system. There are always phenomena which we will not be able to model. Thus, there will always be model errors or model uncertainties. But even if a model describes just a part of the reality it can be very useful for analysis and design, if it describes the dominating dynamic properties of the system. If we feel that the model has the correct structure but that some parameters in the model are imprecise we can try to calculate or estimate them, cj. Chapter 7.

The present chapter describes the basic principles of mathematical modeling. We will see that the models of dynamic systems are in the form of differential equations. This chapter focuses on how to the *develop* dynamic models. Chapter 3 describes other model forms and transformation between model forms. Model analysis is described in the chapters 4, 5 and 6.

2.2 How to Develop Dynamic Models

2.2.1 A Procedure for Mathematical Modeling

Below is described a procedure for developing dynamic mathematical models for physical systems:

1. **Define systems boundaries.** All physical systems works in interaction with other systems. Therefore it is necessary to define the boundaries of the system before we can begin developing a mathematical model for the system, but in most cases defining the boundaries is done quite naturally.
2. **Make simplifying assumptions.** One example is to assume that the temperature in a tank is the same everywhere in their tank, that is, there are homogeneous conditions in the tank.
3. **Use the Balance law for the physical balances in the system, and the define eventual additional conditions.** The Balance law is as follows:

   The rate of change of "material" in the system is equal to net flow of "material" into the system.

   Here "material" is a general term which represent for example energy, mass, momentum, National Product or population. Net flow is the sum of inflow minus the sum of outflow plus generated "material" within the system (for example certain chemical reactions generate energy). The Balance law is illustrated in Figure 2.1.

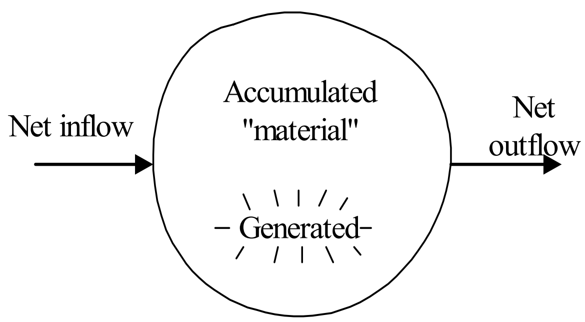

Figure 2.1: Illustration of the Balance law (2.1)

The Balance law can be expressed mathematically as follows:

$$\frac{d(\text{“material”})}{dt} = \sum \text{inflows} - \sum \text{outflows} + \sum \text{generated} \quad (2.1)$$

The Balance law results in one ore more *differential equations* due to the term d/dt.

Additional conditions are in the form of requirements to values of certain variables in the system, e.g. that the mass of liquid in a tank can not be negative.

4. **Present the model on a proper form.** The most common model forms are state-space model, block diagram, transfer function, which are described in Chapter 3, and frequency response, which is described in Chapter 5. The choice of model form depends on the purpose of the model. For example, to develop a simulator for the system, the state-space model is the most convenient model form.

The following sections contains several examples of mathematical modeling. In the examples points 1 and 2 above are applied more or less implicitly.

2.2.2 Mathematical modeling of Mass Systems. Model Terms

This chapter describes mathematical modeling of mass or material systems. Important general terms as *mathematical model*, *input variable*, *output variable*, *model parameter*, and (overall) *block diagram representation* of a system will be introduced in an example of mathematical modeling of a liquid tank (Example 1).

The Balance law (2.1) applied to a mass system becomes a *mass balance*:

$$\frac{dm(t)}{dt} = \sum_i w_i(t) \tag{2.2}$$

where m [kg] is the mass, and w_i [kg/s] is mass inflow (no. i). t [sec] is the time argument.

Example 1 ***Mass balance of a liquid tank***

Figure 2.2 shows a liquid tank with inflow and outflow. The density is the same all over, in the inlet, the outlet, and in the tank. We assume that the tank has straight, vertical walls. The symbols in Figure 2.2 are as follows: q_i is volumetric inflow. q_u is volumetric outflow. h is liquid level. A is cross sectional area. V is liquid volume. m is mass. ρ is density. The flows q_i and q_u and the mass m in the tank are variables. The parameters A and ρ are assumed to be constant.

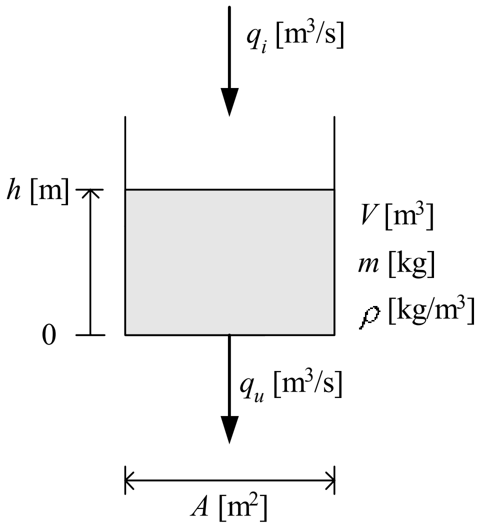

Figure 2.2: Liquid tank

We will develop a mathematical model which expresses how the mass m varies (as a function of time). We write the following mass balance for the mass in the tank:

$$\frac{dm(t)}{dt} = \rho q_i(t) - \rho q_u(t) \tag{2.3}$$

which is a differential equation for m. An additional condition for the differential equation is $m \geq 0$. (2.3) is a *mathematical model* for the system. ρ is a parameter in the model. Parameters are quantities which usually have a constant values and which characterizes the model.

Using the differential equation (2.3) we can calculate $m(t)$ analytically, so that we get a closed formula for $m(t)$, or numerically, using an algorithm for calculating $m(t)$. Such calculations and simulations are described in Chapter 3.

(2.3) is a differential equation for $m(t)$. Perhaps we are more interested in how h will the vary? The correspondence between h and m is a given by

$$m(t) = \rho V(t) = \rho A h(t) \tag{2.4}$$

We insert this into the mass balance (2.3), which then becomes

$$\frac{dm(t)}{dt} = \frac{d\left[\rho V(t)\right]}{dt} = \frac{d\left[\rho A h(t)\right]}{dt} = \rho A \frac{dh(t)}{dt} = \rho q_i(t) - \rho q_u(t) \tag{2.5}$$

where ρ and A the parameters are moved outside the derivation (these are assumed to be constant). We can now cancel ρ and divide by A to get to the following differential equation for $h(t)$:

$$\frac{dh(t)}{dt} = \dot{h}(t) = \frac{1}{A}\left[q_i(t) - q_u(t)\right] \tag{2.6}$$

with the condition $h \geq 0$.

Figure 2.3 shows a *block diagram* for the model (2.6). q_i and q_u are *input variables*, which generally are variables which drive the system (we assume here that q_u is independent of the level, as when it is manipulated using a pump). h is an *output variable*, which generally is the variable which expresses the response or a state of the system. Note that q_u is an input variable despite it represents a physical output (outflow) from the tank!

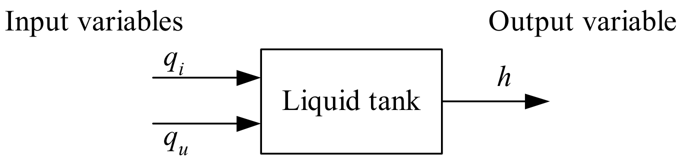

Figure 2.3: Block diagram of a liquid tank

Suppose that the outflow q_u is *dependent* of the level, as when the outflow is through a valve. Suppose that q_u is proportional with the square root of the hydrostatic pressure before the valve:

$$q_u(t) = K_v\sqrt{\rho g h(t)} \tag{2.7}$$

The mass balance becomes

$$\frac{dm(t)}{dt} = \rho q_i(t) - \rho K_v\sqrt{\rho g h(t)} \tag{2.8}$$

Now q_u is no longer a independent input variable to the system. (Instead it is a function of the output variable.)

Let us look at a simulation of the tank. (The simulator is based on the model (2.6) and is implemented in LabVIEW.) Figure 2.4 shows the input signals $q_i(t)$ and $q_u(t)$ and the corresponding time response in the output variable, $h(t)$. The model parameters are shown on the front panel of the simulator (see the figure). The time response is as expected: The level is steadily increasing when the inflow is larger than the outflow, it is constant when the inflow and the outflow are equal, and the level is decreasing when the inflow is smaller than the outflow.

[End of Example 1]

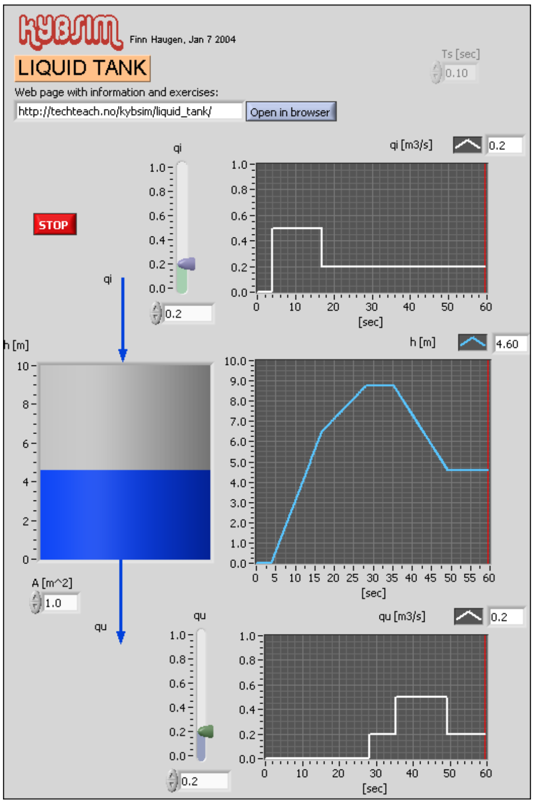

Figure 2.4: Example 1: Simulator of the level $h(t)$ in a liquid tank

Mass balance when the mass is not proportional to the level

in Example 1 the mass m is proportional to the liquid level h, see (2.4). This is because the tank has straight walls. We then ended up with (2.6) as a differential equation for $h(t)$. For example in a spheric tank the mass is not proportional to the liquid level. Let us study this example in detail. The results will however be general. Figure 2.5 shows a spheric tank. We assume that the density ρ is the same all over. The mass balance becomes

$$\frac{dm(t)}{dt} = \frac{d\left[\rho V(t)\right]}{dt} = \rho\frac{d\left[V(t)\right]}{dt} = \rho q_i(t) - \rho q_u(t) \tag{2.9}$$

which, after ρ is cancelled, becomes

$$\frac{dV(t)}{dt} = q_i(t) - q_u(t) \tag{2.10}$$

The volume V is a function of the level h, which he is a function of time t that is, $V = V\left[h(t)\right]$. Using the kernel rule for derivation dV/dt in (2.10)

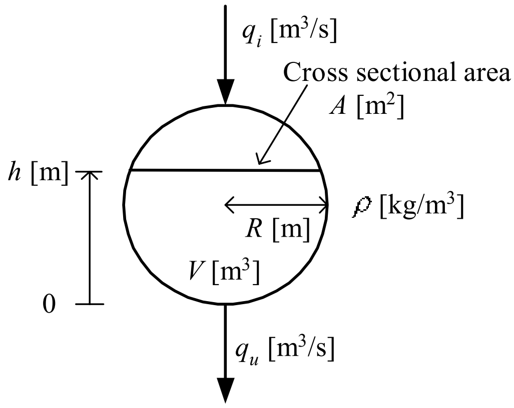

Figure 2.5: A spherical tank, having varying cross sectional area

can be written as

$$\frac{dV(t)}{dt} = \frac{dV\left[h(t)\right]}{dt} = \frac{dV(h)}{dh} \cdot \frac{dh(t)}{dt} \tag{2.11}$$

What is $dV(h)/dh$ in (2.11)? It is the cross sectional area (liquid surface) at the height h. Thus,

$$\frac{dV\left(h\right)}{dh} = A\left[h(t)\right] \tag{2.12}$$

Using (2.11) and (2.12), (2.10) can be written as

$$\frac{dV(t)}{dt} = A\left[h(t)\right] \cdot \frac{dh(t)}{dt} = q_i(t) - q_u(t) \tag{2.13}$$

which also can be written as

$$\frac{dh(t)}{dt} = \dot{h}(t) = \frac{1}{A\left[h(t)\right]}\left[q_i(t) - q_u(t)\right] \tag{2.14}$$

which is on the same form as (2.6). The difference between (2.14) and (2.6) is that A is time varying in the former, while it is constant in the latter.

You certainly wonder what $A\left[h(t)\right]$ is for the spheric liquid tank. Using geometrical considerations in Figure 2.5,

$$A\left[h(t)\right] = \pi\left\{R^2 - \left[h(t) - R\right]^2\right\} \tag{2.15}$$

Let us look at a simulation. The simulator is based on the model (2.14). Figure 2.6 shows $q_i(t)$ and $q_u(t)$ and the corresponding response, $h(t)$. The model parameters are shown on the front panel of the simulator (see the figure). We see that the level varies faster at high level which is because the cross sectional area is relatively small there.

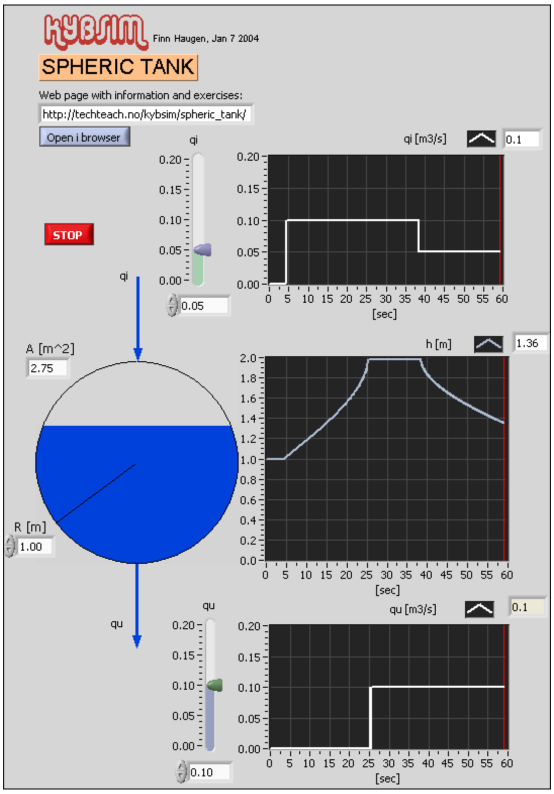

Figure 2.6: Simulation of the liquid level in a spheric tank

Mass balance in the form of mole balance

The mass balance can be in the form of a *mole balance*, as illustrated in Example 2 below.

Example 2 *Mole balance*

Figure 2.7 shows a stirred blending tank where the material A is fed into a tank for blending with a raw material.

The symbols in Figure 2.7 are as follows: V is the liquid volume in the tank. q is the volumetric inflow of the raw material. q is also the volumetric outflow. c_A is the mole density or concentration of material A in the tank. w_A is the mole flow of material A.

We will now develop a mathematical model which expresses how the concentration c_A varies. We make the following assumptions:

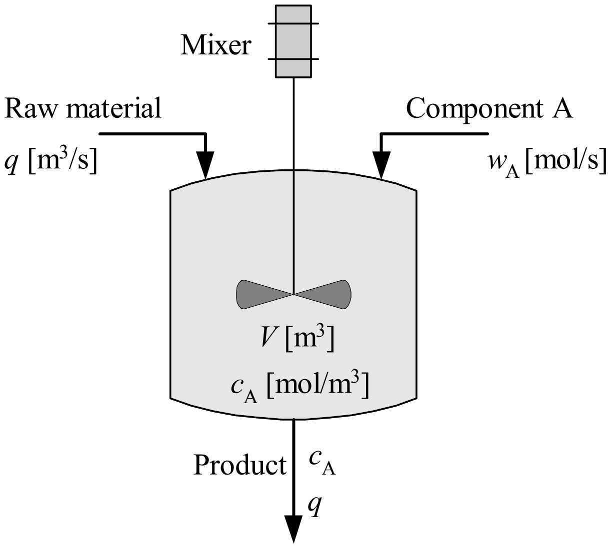

Figure 2.7: Example 2: Blending tank

- The blending in the tank has constant volume.[1]
- The volumetric flow of material A is very small (negligible) compared to the volumetric flow of the raw material.
- There are homogenous conditions (perfect stirring) in the tank.
- The raw material does not contain A.

Mole balance (the total mole number is Vc_A) yields

$$\frac{d\left[Vc_A(t)\right]}{dt} = w_A(t) - c_A(t)q(t) \tag{2.16}$$

By taking V (which is constant) outside the differentiation and then dividing by V on both sides of the equation, we get the following differential equation for c_A:

$$\frac{dc_A(t)}{dt} = \dot{c}_A(t) = \frac{1}{V}\left[w_A(t) - c_A(t)q(t)\right] \tag{2.17}$$

with the condition $c_A \geq 0$.

Figure 2.8 shows a block diagram of the model (2.17). w_A and q are input variables, and c_A is the output variable.

[1]This can be accomplished with for example a level control system.

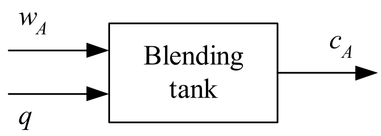

Figure 2.8: Example 2: Block diagram for stirred blending tank

Figure 2.9 shows a simulation of the blending tank. The simulator[2] is based on the model (2.17). The model parameters are shown on the front panel of the simulator (see the figure). We see that the concentration c_A converges to a constant value after the step in w_A.

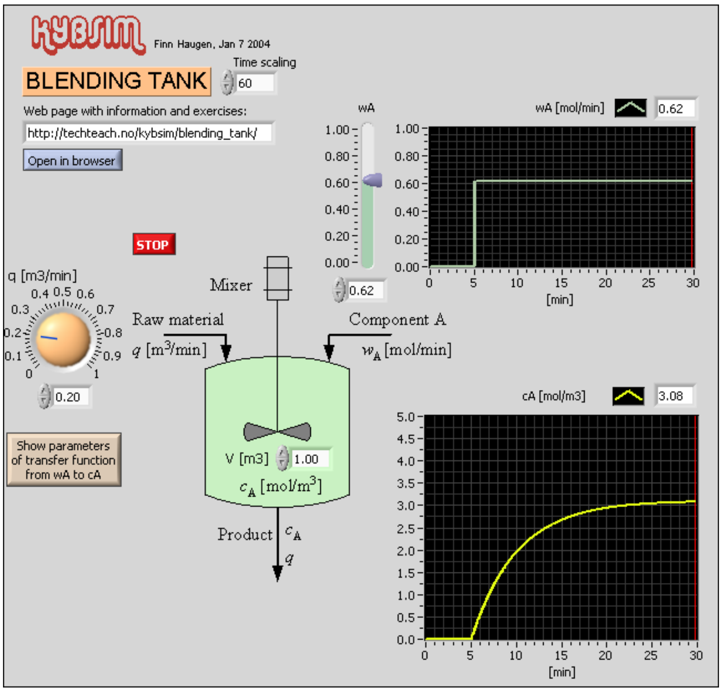

Figure 2.9: Example 2: Simulator for blending tank

[End of Example 2]

[2] which is implemented in LabVIEW

2.2.3 Mathematical Modeling of Thermal Systems

Mathematical modeling of thermal systems is based on the Balance law to set up energy balances. The term *energy* covers temperature-dependent energy, which we can call thermal energy, and kinetic and potential energy. In general we must assume that there is a transformation from one energy form to another within a given system. For example, kinetic energy can be transformed to thermal energy via fiction. For many thermal systems we can assume that the energy consists of only thermal energy and we can neglect the transformation from kinetic and potential energy to thermal energy.

The Balance law (2.1) applied to a thermal system becomes an *energy balance*:

$$\frac{dE(t)}{dt} = \sum_i Q_i(t) \tag{2.18}$$

where E [J] is the thermal energy, and Q_i [J/s] is energy inflow no. i. t [sec] is the time. The energy E is often assumed to be proportional to the temperature and the mass (or volume):

$$E = cmT = c\rho VT = CT \tag{2.19}$$

where T [K] is the temperature, c [J/(kg K)] is specific heat capacity, m [kg] is mass, V [m^3] volume, ρ [kg/m^3] is density, C [J/K] is total heat capacity.

Example 3 ***Heated liquid tank***[3]

Figure 2.10 shows a liquid tank with continuous liquid flow and heat transfer to the environment through the walls. the liquid delivers power through a heating element. P is power from the heating element. T is temperature in the tank and in the outlet flow. T_i he's the temperature in the inlet flow. w is at mass flow. c is specific heat capacity. ρ is density. U is total heat transfer coefficient.

We will now set up a energy balance for the liquid in the tank to find the differential equation which describes the temperature $T(t)$. We will then make the following assumptions:

- The temperature in the liquid in the tank is homogeneous (due to the stirring machine).

[3] This example is used in several sections in later chapters.

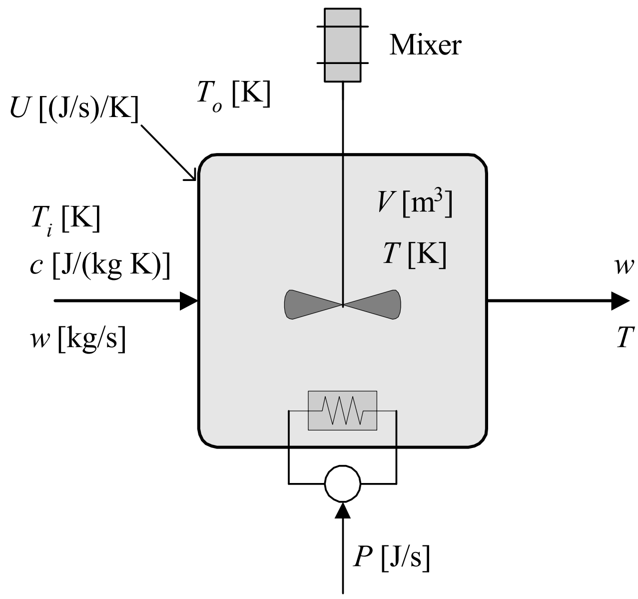

Figure 2.10: Example 3: Heated liquid tank

- The inflow and in the outflow are equal, and the tank is filled by liquid.
- There is no storage of thermal energy in the heating element itself. This means that all of the supplied power to the heating element is supplied (immediately) to the liquid. (Thus, we do not write an energy balance for the heating element.)

The energy balance is based on the following energy transports (power):

1. Power from the heating element:

$$P(t) = Q_1 \tag{2.20}$$

2. Power from the inflow:

$$cw(t)T_i(t) = Q_2 \tag{2.21}$$

3. Power removed via the outflow:

$$-cw(t)T(t) = Q_3 \tag{2.22}$$

4. Power via heat transfer from (or to) the environment:

$$U\left[T_o(t) - T(t)\right] = Q_4 \tag{2.23}$$

The energy balance is

$$\frac{dE(t)}{dt} = Q_1 + Q_2 + Q_3 + Q_4 \tag{2.24}$$

where the energy is given by

$$E(t) = c\rho VT(t)$$

the energy balance can then be written as (here the time argument t is dropped for simplicity):

$$\frac{d\left(c\rho VT\right)}{dt} = P + cwT_i - cwT + U\left(T_o - T\right) \tag{2.25}$$

If we assume that c, ρ and V are constant, we can move $c\rho V$ outside the derivative term. Furthermore, we can combine the the terms on the right side. The result is

$$c\rho V\frac{dT}{dt} = c\rho V\dot{T} = P + cw\left(T_i - T\right) + U\left(T_o - T\right) \tag{2.26}$$

which alternatively can be written

$$\frac{dT}{dt} = \dot{T} = \frac{1}{c\rho V}\left[P + cw\left(T_i - T\right) + U\left(T_o - T\right)\right] \tag{2.27}$$

If the tank is very well isolated, U can be set equal to 0. Then (2.26) becomes

$$c\rho V\dot{T}(t) = P(t) + cw\left[T_i(t) - T(t)\right] \tag{2.28}$$

which can be written

$$\dot{T}(t) = \frac{1}{c\rho V}\left\{P(t) + cw\left[T_i(t) - T(t)\right]\right\} \tag{2.29}$$

Figure 2.11 shows a block diagram of the model (2.28). P and T_i are input variables, and T is an output variable.

Figure 2.12 shows a simulation of the tank. The model parameters are shown on the front panel of the simulator. The simulator is based on the model (2.29). We see that the temperature T converges to a constant value after a positive step in P. We also see that T goes to a constant value after a negative step in T_i.

[End of Example 3]

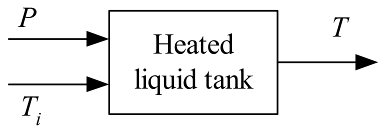

Figure 2.11: Example 3: Block diagram for heated tank